

The Equivalence Problem of Curves in a Riemannian Manifold

M. CASTRILLÓN LÓPEZ

Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM

Departamento de Geometría y Topología

Facultad de Matemáticas, UCM

Avda. Complutense s/n, 28040-Madrid, Spain

E-mail: mcastri@mat.ucm.es

V. FERNÁNDEZ MATEOS*, J. MUÑOZ MASQUÉ

Instituto de Seguridad de la Información, CSIC

C/ Serrano 144, 28006-Madrid, Spain

E-mail: victor.fernandez@iec.csic.es

E-mail: jaime@iec.csic.es

Contents

1	Introduction	2
2	General position	4
2.1	Definitions	4
2.2	Genericity results	5
3	Frenet curves	8
3.1	A Frenet curve defined	8
3.2	Basic formulas	9
3.3	Existence theorems	11
3.4	$\mathcal{F}^{m-1}(M)$ and $\mathcal{N}^{m-1}(M)$	16
4	The equivalence problem	18
4.1	Necessary conditions for congruence	18
4.2	General criterion of congruence	20
4.3	Remarks on the criterion of congruence	21

*The second author is deceased (March 20, 2011). A Memorial Seminar was delivered at Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, on May 13, 2011; also see, <http://www.mat.ucm.es/geomfis/HomenajeVictor.html>. We are proud of having been able to work with such an enthusiastic and generous young researcher as Víctor.

5	Differential invariants	24
5.1	Basic definitions	24
5.2	Stability	26
5.3	Completeness	27
5.4	Generating complete systems of invariants	29
6	Congruence on symmetric manifolds	31
7	Congruence on constant curvature manifolds	32
8	Some examples	34
8.1	Euclidean space	34
8.2	Few isometries	37
8.3	Surfaces	40
8.4	3-dimensional manifolds	41

Abstract

The equivalence problem of curves with values in a Riemannian manifold, is solved. The domain of validity of Frenet's theorem is shown to be the spaces of constant curvature. For a general Riemannian manifold new invariants must thus be added.

There are two important generic classes of curves; namely, Frenet curves and a new class, called curves "in normal position". They coincide in dimensions ≤ 4 only.

A sharp bound for asymptotic stability of differential invariants is obtained, the complete systems of invariants are characterized, and a procedure of generation is presented. Different classes of examples (specially in low dimensions) are analyzed in detail.

Mathematics Subject Classification 2010: Primary: 53A55; Secondary: 53A04, 53B20, 53B21, 53C35, 58A20.

Key words and phrases: Asymptotic stability, complete systems of invariants, congruence, curvatures of a curve, differential invariant, Frenet frame, isometry, Killing vector field, Levi-Civita connection, normal general position, Riemannian metric.

Acknowledgements: Supported by Ministerio de Educación y Ciencia of Spain under grants #MTM2011-22528 and #MTM2010-19111.

1 Introduction

A fundamental problem in Riemannian Geometry is that of equivalence of objects in a determined class, namely, to provide a criterion to know whether two given objects in this class are congruent under isometries or not. Below, this problem is solved in full generality for the simplest case: That of curves with values in a Riemannian manifold.

For the Euclidean space \mathbb{R}^m the equivalence problem is solved by virtue of the Frenet theorem: Two curves parametrized by the arc-length are congruent if and only if they have the same curvatures, $\kappa_1, \dots, \kappa_{m-1}$; but the domain of validity of Frenet's theorem is too restrictive. In fact, Theorem 7.1 states that Frenet's theorem classifies curves in a Riemannian manifold (M, g) if and only if it is of constant curvature. In consequence, in spaces of non-constant curvature new invariants are required (different from curvatures κ_i) to classify curves and, although by means of curvatures a given curve can be reconstructed (see Theorem 3.6), the role of such invariants becomes weaker in spaces of non-constant curvature, even of low dimension (see Theorem 8.13). A generic Riemannian metric in a compact manifold admits no isometry other than the identity map (cf. [7], [8]). Therefore, the difficulty of the equivalence problem is closely related to the size of the isometry group.

Below, the equivalence problem is solved in general by means of functions that are invariant under the isometry group of the Riemannian manifold. In this way, a totally general result is stated for their solution in Theorem 4.4: Essentially, it gives a set of invariants that, together with the classical Frenet curvatures, solves the congruence problem; but it has the inconvenience of using a redundant number of invariants (cf. Remark 4.8, Theorem 8.10). The remarks following its proof (see section 4.3) show, however, that this is the best general result that could be expected. Furthermore, certain classes of Riemannian manifolds can be characterized by means of their invariants; e.g., symmetric spaces (cf. Theorem 6.2) or Lie groups with invariant metrics (cf. Proposition 8.6).

The study of invariants is developed in section 5, where the main questions on such functions are solved for an arbitrary Riemannian manifold: The theorem of asymptotic stability (Theorem 5.4 and Corollary 5.5), the completeness theorem (Theorem 5.9) that allows us to solve the general problem of equivalence by means of a complete system of invariants and the theorem of generation of invariants (Theorem 5.11). An interesting consequence of the generating theorem proves that the ring of invariants can be generated by means of m invariants (where $m = \dim M$) by taking successive total derivatives with respect to t .

In [14] the number of differential invariants with respect to the induced operation of the group G on jet bundles $J^r(\mathbb{R}, G/H)$ of the homogeneous space G/H , is calculated without assuming G is the group of isometries of a metric. According to [20, IV, Example 1.3], if the subgroup H is compact, the quotient manifold G/H admits a Riemannian metric left invariant. The converse statement also holds true, as the isotropy subgroup of a point in a Riemannian manifold is compact (cf. [20, I, Corollary 4.8]). Moreover, it should be noted that most part of the results in [14] hold in general, i.e., without assuming the manifold to be Riemannian homogeneous, as shown in Theorem 5.11 and Remark 5.12 below.

In section 3.3 two basic existence theorems for the generic class \mathcal{F} of Frenet curves are stated. The first result (Theorem 3.6) is a generalization to arbitrary Riemannian manifolds of the existence of curves in Euclidean 3-space with given curvature and torsion, but the second one (Theorem 3.7) is completely new.

In addition to Frenet curves, another generic class \mathcal{N} of curves in a Riemannian

nian manifold is introduced in Definition 2.3, which seems to be the natural setting for the statement of the asymptotic stability theorem (Theorem 5.4). The classes \mathcal{F} and \mathcal{N} are compared in detail in section 3.4. As it is proved in Theorem 3.10, if either $\dim M = m \leq 4$ or g is flat at a neighbourhood of x_0 , then $\mathcal{F}_{t_0, x_0}^{m-1}(M) = \mathcal{N}_{t_0, x_0}^{m-1}(M)$, $(t_0, x_0) \in \mathbb{R} \times M$. In general, however, both generic sets of curves do not coincide, as shown in the examples 3.11 and 3.12.

Finally, we illustrate all these results by studying several important examples in detail: Theorem 8.1 summarizes the structure of invariants on the Euclidean space and, in particular, it shows that the curvatures constitute a basis of invariants on the full set of Frenet curves; in section 8.2 the Riemannian manifolds (M, g) with Killing algebra $\mathfrak{i}(M, g)$ such that $\dim \mathfrak{i}(M, g) \leq \dim M$, are studied; in section 8.3 the case of an arbitrary Riemannian surface is tackled. Theorem 8.10 proves that a finite basis of invariants can efficiently be computed for a generic metric g ; section 8.4 presents the computation of a basis of differential invariants (with their geometric meaning) for Riemannian homogeneous complete 3-dimensional manifolds, according the dimension of their Killing algebra be of dimension 4 or 3, as the solution to the equivalent problem is completely different in both cases. As a by-product of the results obtained above a explicit geometric description of invariant functions in complete Riemannian manifolds of dimensions 2 and 3 is provided.

2 General position

2.1 Definitions

Definition 2.1. A smooth curve $\sigma: (a, b) \rightarrow M$ taking values into a manifold M endowed with a linear connection ∇ is said to be in *general position up to the order r* , for $1 \leq r \leq m = \dim M$, at $t_0 \in (a, b)$ if the vector fields $T^\sigma, \nabla_{T^\sigma} T^\sigma, \dots, \nabla_{T^\sigma}^{r-1} T^\sigma$ along σ are linearly independent at t_0 , where T^σ is the tangent field to σ . The curve σ is in general position up to the order r if it is in general position up to this order for every $t \in (a, b)$.

Geometrically, a curve in general position is as twisted as possible. For example, if (M, g) is of constant curvature, then σ is in general position up to order r at $\sigma(t_0)$ if and only if no neighbourhood $\{\sigma(t) : |t - t_0| < \varepsilon\}$ is contained into an auto-parallel submanifold (cf. [20, VII, Section 8]) of M of dimension $< r$. The condition of being in general position up to first order is none other than an immersion and hence, it is independent of ∇ ; but for $r \geq 2$ the condition of being in general position up to order r does depend on ∇ .

Lemma 2.2. *Let $\sigma: (a, b) \rightarrow M$ be a smooth curve taking values into a manifold M endowed with a linear connection ∇ . If (x^1, \dots, x^m) is a normal coordinate system with respect to ∇ centered at $x_0 = \sigma(t_0)$, $a < t_0 < b$, then the tangent vectors*

$$(1) \quad U_{t_0}^{\sigma, k} = \frac{d^k(x^i \circ \sigma)}{dt^k}(t_0) \frac{\partial}{\partial x^i} \Big|_{\sigma(t_0)} \in T_{\sigma(t_0)} M, \quad k \geq 1, k \in \mathbb{N},$$

do not depend on the particular normal coordinates chosen.

Proof. If $x^i = a_j^i x^j$, $A = (a_j^i) \in Gl(m, \mathbb{R})$, is another normal coordinate system, then

$$\frac{\partial}{\partial x'^i} = b_i^h \frac{\partial}{\partial x^h}, \quad (b_i^h) = A^{-1},$$

and hence

$$\begin{aligned} \frac{d^k(x^i \circ \sigma)}{dt^k}(t_0) \frac{\partial}{\partial x'^i} \Big|_{\sigma(t_0)} &= \frac{d^k(a_j^i x^j \circ \sigma)}{dt^k}(t_0) b_i^h \frac{\partial}{\partial x^h} \Big|_{\sigma(t_0)} \\ &= b_i^h a_j^i \frac{d^k(x^j \circ \sigma)}{dt^k}(t_0) \frac{\partial}{\partial x^h} \Big|_{\sigma(t_0)} \\ &= \frac{d^k(x^j \circ \sigma)}{dt^k}(t_0) \frac{\partial}{\partial x^j} \Big|_{\sigma(t_0)}. \end{aligned}$$

□

Definition 2.3. A curve σ is said to be in *normal general position up to the order r* at $t_0 \in (a, b)$ if the tangent vectors $U_{t_0}^{\sigma,1}, U_{t_0}^{\sigma,2}, \dots, U_{t_0}^{\sigma,r}$ are linearly independent. The curve σ is in normal general position up to the order r if it is in normal general position up to this order for every $t \in (a, b)$.

2.2 Genericity results

Lemma 2.4. Let $(U; x^1, \dots, x^m)$ be a coordinate open domain in a smooth manifold M endowed with a linear connection ∇ . There exist smooth functions

$$F^{k,i} : J^k(\mathbb{R}, U) \rightarrow \mathbb{R}, \quad k \in \mathbb{N}, 1 \leq i \leq m,$$

such that,

$$(2) \quad (\nabla_{T^\sigma}^k T^\sigma)_t = \left(\frac{d^{k+1}(x^i \circ \sigma)}{dt^{k+1}}(t) + F^{k,i}(j_t^k \sigma) \right) \frac{\partial}{\partial x^i} \Big|_{\sigma(t)},$$

for every curve $\sigma : \mathbb{R} \rightarrow U$ and every $t \in \mathbb{R}$, which are determined as follows:

$$(3) \quad F^{0,i} = 0,$$

$$(4) \quad F^{1,i} = \sum_{j,h=1}^m \Gamma_{jh}^i x_1^j x_1^h,$$

where Γ_{jh}^i are the local symbols of ∇ in $(U; x^1, \dots, x^m)$, and

$$(5) \quad F^{k,i} = D_t(F^{k-1,i}) + \sum_{h,j=1}^m \Gamma_{hj}^i x_1^j (x_k^h + F_h^{k-1}), \quad \forall k \geq 2,$$

$(x_l^h)_{0 \leq l \leq k}^{1 \leq h \leq m}$ being the coordinates induced by $(x^i)_{i=1}^m$ in the k -jet bundle, i.e.,

$$x_l^h(j_t^k \sigma) = \frac{d^l(x^h \circ \sigma)}{dt^l}(t), \quad x_0^h = x^h, \quad 0 \leq l \leq k, \quad 1 \leq h \leq m,$$

and D_t denotes the “total derivative” with respect to t , namely,

$$D_t = \frac{\partial}{\partial t} + \sum_{r=0}^{\infty} x_{r+1}^i \frac{\partial}{\partial x_r^i}.$$

Proposition 2.5. *Let M be a smooth manifold of dimension m endowed with a linear connection ∇ . The set of curves in general position up to the order $r \leq m-1$ is a dense open subset in $C^\infty(\mathbb{R}, M)$ with respect to the strong topology.*

Proof. By using the formulas (2) it follows that the mapping

$$(6) \quad \begin{aligned} \Phi_\nabla^r: J^r(\mathbb{R}, M) &\rightarrow \mathbb{R} \times (\oplus^r TM), \\ \Phi_\nabla^r(j_{t_0}^r \sigma) &= \left(t_0; T_{t_0}^\sigma, (\nabla_{T^\sigma} T^\sigma)_{t_0}, \dots, (\nabla_{T^\sigma}^{r-1} T^\sigma)_{t_0} \right), \end{aligned}$$

is a diffeomorphism inducing the identity on $J^0(\mathbb{R}, M) = \mathbb{R} \times M$. We set

$$E = \{ (t, X^1, \dots, X^r) \in \mathbb{R} \times (\oplus^r TM) : X^1 \wedge \dots \wedge X^r = 0 \},$$

for every $1 \leq k \leq r-1$ and for every strictly increasing system of indices $1 \leq i_1 < \dots < i_k \leq r$ we set

$$E_{i_1, \dots, i_k} = \left\{ (t, X^1, \dots, X^r) \in \mathbb{R} \times (\oplus^r TM) : \begin{aligned} &X^{i_1} \wedge \dots \wedge X^{i_k} \neq 0, \\ &X^{j_1}, \dots, X^{j_{r-k}} \in \langle X^{i_1}, \dots, X^{i_k} \rangle, \end{aligned} \right\}$$

with $j_1 < \dots < j_{r-k}$ and $\{j_1, \dots, j_{r-k}\} = \{1, 2, \dots, r\} \setminus \{i_1, \dots, i_k\}$, and finally we set $E_0 = \mathbb{R} \times Z$, Z being the zero section in $\oplus^r TM$. Hence

$$E = E_0 \cup \bigcup_{k=1}^{r-1} \bigcup_{i_1 < \dots < i_k} E_{i_1, \dots, i_k}.$$

Moreover, if $U_k(M) \subset \oplus^k TM$ denotes the open subset of all linearly independent systems of k vectors, then the mapping

$$\begin{aligned} A_{i_1, \dots, i_k}: \mathbb{R}^{1+k(r-k)} \times U_k(M) &\rightarrow \mathbb{R} \times (\oplus^r TM), \\ A_{i_1, \dots, i_k}(t; \lambda_1^1, \dots, \lambda_k^1, \dots, \lambda_1^{r-k}, \dots, \lambda_k^{r-k}; X^1, \dots, X^r) &= (t; \bar{X}^1, \dots, \bar{X}^r), \\ \bar{X}^{i_h} &= X^h, \quad 1 \leq h \leq k, \\ \bar{X}^{j_h} &= \sum_{i=1}^k \lambda_i^h X^i, \quad 1 \leq h \leq r-k, \end{aligned}$$

is an injective immersion such that $\text{im}(A_{i_1, \dots, i_k}) = E_{i_1, \dots, i_k}$, and we have

$$\begin{aligned} \text{codim } E_{i_1, \dots, i_k} &= \dim(\mathbb{R} \times (\oplus^r TM)) - \dim E_{i_1, \dots, i_k} \\ &= (1 + m + rm) - (1 + k(r-k) + m + km) \\ &= (m-k)(r-k) \\ &\geq m+1-r, \end{aligned}$$

as the product $(m-k)(r-k)$ takes its minimum value when k takes its maximum value, i.e., $k = r-1$. Accordingly, $Y_{i_1, \dots, i_k} = (\Phi_\nabla^r)^{-1}(E_{i_1, \dots, i_k})$ is a submanifold in $J^r(\mathbb{R}, M)$ of codimension $(m-k)(r-k)$. From Thom's transversality theorem (e.g., see [35, VII, Théorème 4.2]) the set of curves $\sigma: \mathbb{R} \rightarrow M$ the r -jet extension of which, $j^r \sigma$, is transversal to Y_{i_1, \dots, i_k} is a residual subset (and hence dense) in $C^\infty(\mathbb{R}, M)$ for the strong topology. For such curves, $(j^r \sigma)^{-1}(Y_{i_1, \dots, i_k})$ is a submanifold of the real line of codimension $(m-k)(r-k) \geq m+1-r$. If $r \leq m-1$, then it is only possible if such a submanifold is the empty set. Consequently, for $r \leq m-1$, the following formula holds:

$$\begin{aligned} F^r &= \{\sigma \in C^\infty(\mathbb{R}, M) : j^r \sigma \text{ is transversal to every } Y_{i_1, \dots, i_k}\} \\ &= \{\sigma \in C^\infty(\mathbb{R}, M) : (j^r \sigma)(\mathbb{R}) \cap Y = \emptyset\}, \end{aligned}$$

where $Y = Y_0 \cup \bigcup_{k=1}^{r-1} \bigcup_{i_1 < \dots < i_k} Y_{i_1, \dots, i_k}$. Therefore $\Phi_\nabla^r(j^r \sigma(\mathbb{R})) \cap E = \emptyset$ if $\sigma \in F^r$; in other words, σ is a curve in general position up to order r with respect to ∇ .

Finally, we prove that F^r is an open subset. If d is a complete distance function defining the topology in $J^r(\mathbb{R}, M)$, then for every $\sigma \in F^r$ the function $\delta_\sigma: \mathbb{R} \rightarrow \mathbb{R}^+$, $\delta_\sigma(t) = d(j_t^r \sigma, Y) > 0$ makes sense as Y is a closed subset and

$$N(\sigma) = \{\gamma \in C^\infty(\mathbb{R}, M) : d(j_t^r \sigma, j_t^r \gamma) < \delta_\sigma(t), \forall t \in \mathbb{R}\}$$

is a neighbourhood of σ in the strong topology of order r and hence, also in the strong topology of order ∞ . As $\gamma \in N(\sigma)$ implies $\gamma \in F^r$, we can conclude. \square

Remark 2.6. The statement of Proposition 2.5 is the best possible, as the curves in general position up to the order $m = \dim M$ with respect to a linear connection ∇ are not dense in $C^\infty(\mathbb{R}, M)$ for the strong topology, because inflection points are unavoidable. In fact, with the similar notations as in the proof of Proposition 2.5, we set

$$\begin{aligned} F^m &= \{\sigma \in C^\infty(\mathbb{R}, M) : j^m \sigma(\mathbb{R}) \cap Y = \emptyset\}, \\ \bar{F}^m &= \{\sigma \in C^\infty(\mathbb{R}, M) : j^m \sigma \text{ is transversal to every } Y_{i_1, \dots, i_k}\}. \end{aligned}$$

The set \bar{F}^m is dense in $C^\infty(\mathbb{R}, M)$ as it is residual and F^m coincides with the set of curves in general position up to order m . In order to prove that F^m is not dense, it suffices to obtain an open subset contained in its complementary set. We set

$$Y' = \bigcup_{k=0}^{m-2} \bigcup_{i_1 < \dots < i_k} Y_{i_1, \dots, i_k}; \quad Y_i = (\Phi_\nabla^m)^{-1}(E_i), \quad 1 \leq i \leq m,$$

where E_i is the set of points $(t, X^1, \dots, X^m) \in \mathbb{R} \times (\oplus^m TM)$ such that,

- i) $X^1 \wedge \dots \wedge \widehat{X^i} \wedge \dots \wedge X^m \neq 0$,
- ii) $X^i \in \langle X^1, \dots, \widehat{X^i}, \dots, X^m \rangle$.

Then, $Y = Y' \cup Y_1 \cup \dots \cup Y_m$ and $Y_0 = Y_1 \cap \dots \cap Y_m$ is an open subset in each Y_i ; hence Y_0 is a submanifold of codimension 1 in $J^m(\mathbb{R}, M)$. According to a classical result (see [27, Theorem 6.1]) there exists a curve $\sigma: \mathbb{R} \rightarrow M$ such that, 1) $j^m\sigma$ is transversal to Y_0 , and 2) $j^m\sigma(\mathbb{R}) \cap Y_0 \neq \emptyset$. Therefore, $j^m\sigma(\mathbb{R}) \cap Y \neq \emptyset$. Moreover, according to [25, Lemma 1, p. 45], given a neighbourhood U of t , there exists a neighbourhood E_σ of σ in the weak (and hence, in the strong) topology, such that $\tau \in E_\sigma$ implies $j^m\tau$ cuts transversally to Y at some point $t' \in U$. Hence, $\tau \in E_\sigma$ implies $j^m\tau(\mathbb{R}) \cap Y \neq \emptyset$, i.e., $\tau \notin F^m$, and σ thus possesses a neighbourhood of curves not belonging to F^m .

Proposition 2.7. *Let M be a smooth manifold of dimension m endowed with a linear connection ∇ . The set of curves in normal general position up to the order $r \leq m - 1$ is a dense open subset in $C^\infty(\mathbb{R}, M)$ with respect to the strong topology.*

Proof. It is similar to the proof of Proposition 2.5 by using the fact that the mapping

$$\begin{aligned} \Psi_\nabla^r: J^r(\mathbb{R}, M) &\rightarrow \mathbb{R} \times (\oplus^r TM), \\ \Psi_\nabla^r(j_{t_0}^r \sigma) &= (t_0; U_{t_0}^{1,\sigma}, U_{t_0}^{2,\sigma}, \dots, U_{t_0}^{r,\sigma}), \end{aligned}$$

is a diffeomorphism over $\mathbb{R} \times M$. □

3 Frenet curves

3.1 A Frenet curve defined

Definition 3.1. A curve $\sigma: (a, b) \rightarrow M$ with values into a Riemannian manifold (M, g) is said to be a *Frenet curve* if σ is in general position up to order $m - 1$ with respect to the Levi-Civita connection of the metric g .

Proposition 3.2 (Frenet frame, [3], [11], [12], [15], [18], [32]). *If (M, g) is an oriented connected Riemannian manifold of dimension m and $\sigma: (a, b) \rightarrow M$ is a Frenet curve, then there exist unique vector fields $X_1^\sigma, \dots, X_m^\sigma$ defined along σ and smooth functions $\kappa_0^\sigma, \dots, \kappa_{m-1}^\sigma: (a, b) \rightarrow \mathbb{R}$ with $\kappa_j^\sigma > 0$, $0 \leq j \leq m - 2$, such that,*

- (i) $(X_1^\sigma(t), \dots, X_m^\sigma(t))$ is a positively oriented orthonormal linear frame, $\forall t \in (a, b)$.
- (ii) The systems $(X_1^\sigma(t), \dots, X_i^\sigma(t))$, $(T_t^\sigma, (\nabla_{T^\sigma} T^\sigma)_t, \dots, (\nabla_{T^\sigma}^{i-1} T^\sigma)_t)$ span the same vector subspace and they are equally oriented for every $1 \leq i \leq m - 1$ and every $t \in (a, b)$.

(iii) The following formulas hold:

- (a) $T^\sigma = \kappa_0^\sigma X_1$,
- (b) $\nabla_{X_1^\sigma} X_1^\sigma = \kappa_1^\sigma X_2^\sigma$,

$$\begin{aligned}
(c) \quad \nabla_{X_1^\sigma} X_i^\sigma &= -\kappa_{i-1}^\sigma X_{i-1}^\sigma + \kappa_i^\sigma X_{i+1}^\sigma, \quad 2 \leq i \leq m-1, \\
(d) \quad \nabla_{X_1^\sigma} X_m^\sigma &= -\kappa_{m-1}^\sigma X_{m-1}^\sigma.
\end{aligned}$$

Definition 3.3. The frame $(X_1^\sigma, \dots, X_m^\sigma)$ along σ determined by the conditions (i)-(iii) above is called the *Frenet frame* of σ , and the functions $\kappa_0^\sigma, \dots, \kappa_{m-1}^\sigma$ are the *curvatures* of σ .

3.2 Basic formulas

According to the item (ii) of Proposition 3.2 there exist functions $f_{ij}^\sigma \in C^\infty(a, b)$, $1 \leq i \leq j \leq m$, such that,

$$(7) \quad \nabla_{T^\sigma}^{j-1} T^\sigma = \sum_{i=1}^j f_{ij}^\sigma X_i^\sigma, \quad 1 \leq j \leq m,$$

and by using the equations (a)-(d) in the item (iii) above the following recurrence formulas are obtained for these functions:

$$\begin{aligned}
(8) \quad & \begin{cases} f_{11}^\sigma = \kappa_0^\sigma, \\ f_{12}^\sigma = \frac{df_{11}^\sigma}{dt}, \\ f_{22}^\sigma = f_{11}^\sigma \kappa_0^\sigma \kappa_1^\sigma, \end{cases} \\
(9) \quad 3 \leq j \leq m \quad & \begin{cases} f_{1j}^\sigma = \frac{df_{1,j-1}^\sigma}{dt} - f_{2,j-1}^\sigma \kappa_0^\sigma \kappa_1^\sigma, \\ f_{ij}^\sigma = \frac{df_{i,j-1}^\sigma}{dt} - f_{i+1,j-1}^\sigma \kappa_0^\sigma \kappa_i^\sigma + f_{i-1,j-1}^\sigma \kappa_0^\sigma \kappa_{i-1}^\sigma, \\ 2 \leq i \leq j-2, \\ f_{j-1,j}^\sigma = \frac{df_{j-1,j-1}^\sigma}{dt} + f_{j-2,j-1}^\sigma \kappa_0^\sigma \kappa_{j-2}^\sigma, \\ f_{jj}^\sigma = f_{j-1,j-1}^\sigma \kappa_0^\sigma \kappa_{j-1}^\sigma. \end{cases}
\end{aligned}$$

Proposition 3.4. If $\sigma: (a, b) \rightarrow M$ is a Frenet curve in an oriented connected Riemannian manifold (M, g) , then

$$\begin{aligned}
\kappa_0^\sigma &= \sqrt{\Delta_1^\sigma}, \\
\kappa_1^\sigma &= \sqrt{\frac{\Delta_2^\sigma}{(\Delta_1^\sigma)^3}}, \\
\kappa_i^\sigma &= \frac{\varepsilon_i \sqrt{\Delta_{i-1}^\sigma \Delta_{i+1}^\sigma}}{\sqrt{\Delta_1^\sigma \Delta_i^\sigma}}, \quad 2 \leq i \leq m-1,
\end{aligned}$$

where

$$(10) \quad \begin{cases} \Delta_k^\sigma = \det \left(g \left(\nabla_{T^\sigma}^{i-1} T^\sigma, \nabla_{T^\sigma}^{j-1} T^\sigma \right) \right)_{i,j=1}^k, \\ \varepsilon_i = 1 \text{ for } 2 \leq i \leq m-2, \text{ and } \varepsilon_{m-1} = \pm 1. \end{cases}$$

Proof. The formulas in the statement follow from (7), (8), and (9) taking the identity

$$\begin{aligned}\Delta_k^\sigma &= \left(\det (f_{ij}^\sigma)_{i,j=1}^k \right)^2 \\ &= \prod_{i=1}^k (f_{ii}^\sigma)^2\end{aligned}$$

into account. \square

For a smooth curve σ , the property of being a Frenet curve at t depends on $j_t^{m-1}\sigma$ only; hence for every $t \in \mathbb{R}$ we can speak about the open subset $\mathcal{F}_t^{m-1}(M) \subset J_t^{m-1}(\mathbb{R}, M)$ of Frenet jets. Let

$$f_{ij}: (\pi_{m-1}^m)^{-1}(\mathcal{F}^{m-1}(M)) \rightarrow \mathbb{R}, \quad 1 \leq i \leq j \leq m,$$

be the mapping defined by $f_{ij}(j_t^m \sigma) = f_{ij}^\sigma(t)$, $\pi_l^k: J^k(\mathbb{R}, M) \rightarrow J^l(\mathbb{R}, M)$, $k \geq l$, being the canonical projections. Similarly, let

$$(11) \quad \Delta_k: J^k(\mathbb{R}, M) \rightarrow \mathbb{R}, \quad 1 \leq k \leq m,$$

be the mapping given by $\Delta_k(j_t^k \sigma) = \Delta_k^\sigma(t)$, which is well defined according to the formula (10).

Proposition 3.5. *If $\sigma: (a, b) \rightarrow M$, $\bar{\sigma}: (a, b) \rightarrow \bar{M}$ are two Frenet curves with values in Riemannian manifolds (M, g) , (\bar{M}, \bar{g}) , then $|\kappa_i^\sigma| = |\kappa_i^{\bar{\sigma}}|$, $0 \leq i \leq m-1$, if and only if,*

$$(12) \quad g\left(\nabla_{T^\sigma}^{i-1} T^\sigma, \nabla_{T^\sigma}^{j-1} T^\sigma\right) = \bar{g}\left(\bar{\nabla}_{T^{\bar{\sigma}}}^{i-1} T^{\bar{\sigma}}, \bar{\nabla}_{T^{\bar{\sigma}}}^{j-1} T^{\bar{\sigma}}\right), \quad i, j = 1, \dots, m,$$

$\nabla, \bar{\nabla}$ being the Levi-Civita connections associated to g, \bar{g} .

Proof. If (12) holds for every $i, j = 1, \dots, m$, then $\Delta_k^\sigma = \Delta_k^{\bar{\sigma}}$ for $1 \leq k \leq m$. From the formulas (10) in Proposition 3.4 we deduce $\kappa_i^\sigma = \kappa_i^{\bar{\sigma}}$ for $0 \leq i \leq m-2$ and $|\kappa_{m-1}^\sigma| = |\kappa_{m-1}^{\bar{\sigma}}|$. Conversely, if $|\kappa_i^\sigma| = |\kappa_i^{\bar{\sigma}}|$, $0 \leq i \leq m-1$, then from the formulas (8) and (9) by recurrence on the subindex of κ_i we obtain $|f_{mm}^\sigma| = |f_{mm}^{\bar{\sigma}}|$ and $f_{ij}^\sigma = f_{ij}^{\bar{\sigma}}$ otherwise. Hence, for every $i, j = 1, \dots, m$, we have

$$\begin{aligned}g\left(\nabla_{T^\sigma}^{i-1} T^\sigma, \nabla_{T^\sigma}^{j-1} T^\sigma\right) &= \sum_{k=1}^i \sum_{l=1}^j f_{ki}^\sigma f_{lj}^\sigma \delta_{kl} \\ &= \sum_{k=1}^i \sum_{l=1}^j f_{ki}^{\bar{\sigma}} f_{lj}^{\bar{\sigma}} \delta_{kl} \\ &= \bar{g}\left(\bar{\nabla}_{T^{\bar{\sigma}}}^{i-1} T^{\bar{\sigma}}, \bar{\nabla}_{T^{\bar{\sigma}}}^{j-1} T^{\bar{\sigma}}\right).\end{aligned}$$

\square

3.3 Existence theorems

The fundamental theorem for curves in Euclidean 3-space (cf. [33, pp. 29–31]) states that if two smooth functions $\kappa(s) > 0$, $\tau(s)$ are given, then there exists a unique curve for which s is the arc length, κ the curvature, and τ the torsion, the moving trihedron of which at $s = s_0$ coincides with the coordinate axes. The full generalization of this result is as follows:

Theorem 3.6. *Let (M, g) be an m -dimensional oriented Riemannian manifold and let (v_1, \dots, v_m) be a positively oriented orthonormal basis for $T_{x_0}M$. Given functions $\kappa_j \in C^\infty(t_0 - \delta, t_0 + \delta)$, $0 \leq j \leq m-1$, with $\kappa_j > 0$ for $0 \leq j \leq m-2$, there exists $0 < \varepsilon \leq \delta$ and a unique Frenet curve $\sigma: (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow M$ such that,*

- (i) $\sigma(t_0) = x_0$,
- (ii) $X_j^\sigma(t_0) = v_j$ for $1 \leq j \leq m$,
- (iii) $\kappa_j^\sigma = \kappa_j$ for $0 \leq j \leq m-1$.

Proof. Let $(U; x^1, \dots, x^m)$ be the normal coordinate system centered at x_0 associated to the orthonormal linear frame (v_1, \dots, v_m) given in the statement, let $p_M^m: \oplus^m TM \rightarrow M$ be the bundle projection, and let denote by (x^i, y_k^j) , $i, j, k = 1, \dots, m$, the induced coordinate system on $(p_M^m)^{-1}(U)$, i.e.,

$$u_j = y_j^i(u) \frac{\partial}{\partial x^i} \Big|_x, \quad \forall u = (u_1, \dots, u_m) \in \oplus^m T_x M, \quad x \in U.$$

First of all, we prove that the Frenet formulas are locally equivalent to a system of first-order ordinary differential equations on the manifold $\oplus^m TM$. In fact, as a computation shows, the formulas (a)-(d) in Proposition 3.2 can be written in local coordinates as follows:

$$(13) \quad \frac{d(x^j \circ \sigma)}{dt} = \kappa_0^\sigma (y_1^j \circ X^\sigma),$$

$$(14) \quad \frac{d(y_1^j \circ X^\sigma)}{dt} = \kappa_0^\sigma \kappa_1^\sigma (y_2^j \circ X^\sigma) - \kappa_0^\sigma (\Gamma_{hi}^j \circ \sigma) (y_1^h \circ X^\sigma) (y_1^i \circ X^\sigma),$$

$$(15) \quad \begin{cases} \frac{d(y_i^c \circ X^\sigma)}{dt} = \kappa_0^\sigma [\kappa_i^\sigma (y_{i+1}^c \circ X^\sigma) - \kappa_{i-1}^\sigma (y_{i-1}^c \circ X^\sigma)] \\ \quad - \kappa_0^\sigma (\Gamma_{ab}^c \circ \sigma) (y_1^a \circ X^\sigma) (y_1^b \circ X^\sigma), \quad 2 \leq i \leq m-1, \end{cases}$$

$$(16) \quad \frac{d(y_m^c \circ X^\sigma)}{dt} = -\kappa_0^\sigma \kappa_{m-1}^\sigma (y_{m-1}^c \circ X^\sigma) - \kappa_0^\sigma (\Gamma_{ab}^c \circ \sigma) (y_1^a \circ X^\sigma) (y_m^b \circ X^\sigma),$$

where Γ_{ab}^c are the components of the Levi-Civita connection ∇ of g with respect to the coordinate system $(x^h)_{h=1}^m$ and $X^\sigma: (a, b) \rightarrow \oplus^m TM$, $a < t_0 < b$, is the curve given by $X^\sigma(t) = t(X_1^\sigma(t), \dots, X_m^\sigma(t))$, $\forall t \in (a, b)$.

Hence the functions $x^h \circ \sigma, y_j^i \circ X^\sigma: (a, b) \rightarrow \mathbb{R}$, $h, i, j = 1, \dots, m$, are the only solutions to the system (13)–(16) satisfying the initial conditions (i), (ii) in the statement; i.e., $(x^h \circ \sigma)(t_0) = x^h(x_0)$, $(y_j^i \circ X^\sigma)(t_0) = y_j^i(v_1, \dots, v_m) = \delta_j^i$.

Conversely, if X^σ and the curvatures κ_{j-1}^σ , $1 \leq j \leq m$, are replaced by an arbitrary smooth curve $X = (X_1, \dots, X_m): (t_0 - \delta, t_0 + \delta) \rightarrow \oplus^m TM$, and the given functions κ_{j-1} , $1 \leq j \leq m$, respectively, with $\sigma = p_M^m \circ X$, into the equations (13)–(16) above, then the following system is obtained:

$$(17) \quad \frac{d(x^j \circ \sigma)}{dt} = \kappa_0(y_1^j \circ X),$$

$$(18) \quad \frac{d(y_1^j \circ X)}{dt} = \kappa_0 \kappa_1(y_2^j \circ X) - \kappa_0(\Gamma_{hi}^j \circ \sigma)(y_1^h \circ X)(y_1^i \circ X),$$

$$(19) \quad \begin{cases} \frac{d(y_i^c \circ X)}{dt} = \kappa_0[\kappa_i(y_{i+1}^c \circ X) - \kappa_{i-1}(y_{i-1}^c \circ X)] \\ \quad - \kappa_0(\Gamma_{ab}^c \circ \sigma)(y_1^a \circ X)(y_i^b \circ X), \quad 2 \leq i \leq m-1, \end{cases}$$

$$(20) \quad \frac{d(y_m^c \circ X)}{dt} = -\kappa_0 \kappa_{m-1}(y_{m-1}^c \circ X) - \kappa_0(\Gamma_{ab}^c \circ \sigma)(y_1^a \circ X)(y_m^b \circ X),$$

We claim that the only solution $x^h \circ \sigma, y_j^i \circ X: (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow \mathbb{R}$ to the system (17)–(20) satisfying the initial conditions

$$\begin{aligned} (x^h \circ \sigma)(t_0) &= x^h(x_0), \quad 1 \leq h \leq m, \\ (y_j^i \circ X)(t_0) &= \delta_j^i, \quad i, j = 1, \dots, m, \end{aligned}$$

provides the desired Frenet curve.

First, we observe that from the very definition of (17)–(20), the linear frame (X_1, \dots, X_m) —defined along the curve σ with components $x^h \circ \sigma$ —determined by X , i.e., $X_j = (y_j^i \circ X) \frac{\partial}{\partial x^i}$, satisfies the following equations:

$$(21) \quad \begin{cases} T^\sigma = \kappa_0 X_1, \\ \nabla_{X_1} X_1 = \kappa_1 X_2, \\ \nabla_{X_1} X_i = -\kappa_{i-1} X_{i-1} + \kappa_i X_{i+1}, \quad 2 \leq i \leq m-1, \\ \nabla_{X_1} X_m = -\kappa_{m-1} X_{m-1}. \end{cases}$$

Next, the item (i) in Proposition 3.2 is proved to hold for this linear frame. In fact, the functions $\varphi_{ij}(t) = g(X_i(t), X_j(t))$, $|t - t_0| < \varepsilon$, $1 \leq i \leq j \leq m$, are the

only solution to the system

$$\begin{aligned}
\frac{d\varphi_{11}}{dt} &= 2\kappa_0\kappa_1\varphi_{12}, \\
\frac{d\varphi_{1j}}{dt} &= \kappa_0(\kappa_1\varphi_{2j} - \kappa_{j-1}\varphi_{1,j-1} + \kappa_j\varphi_{1,j+1}), \\
2 \leq j &\leq m-1, \\
\frac{d\varphi_{1m}}{dt} &= \kappa_0(\kappa_1\varphi_{2m} - \kappa_{m-1}\varphi_{2,m-1}), \\
\frac{d\varphi_{ij}}{dt} &= \kappa_0(\kappa_i\varphi_{i+1,j} + \kappa_j\varphi_{i,j+1} - \kappa_{i-1}\varphi_{i-1,j} - \kappa_{j-1}\varphi_{i,j-1}), \\
2 \leq i &\leq j \leq m-1, \\
\frac{d\varphi_{im}}{dt} &= \kappa_0(\kappa_i\varphi_{i+1,m} - \kappa_{i-1}\varphi_{i-1,m} - \kappa_{m-1}\varphi_{i,m-1}), \\
2 \leq i &\leq m-1, \\
\frac{d\varphi_{mm}}{dt} &= -2\kappa_0\kappa_{m-1}\varphi_{m-1,m},
\end{aligned}$$

such that $\varphi_{ij}(0) = \delta_{ij}$, but Kronecker deltas are readily seen to be also a solution to this system; hence $g(X_i(t), X_j(t)) = \delta_{ij}$. By virtue of the assumption, one has $\text{vol}_g(X_1(0), \dots, X_m(0)) = \text{vol}_g(v_1, \dots, v_m) = 1$, and accordingly, $\text{vol}_g(X_1(t), \dots, X_m(t)) = 1$ for every $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$.

Finally, as the curvatures κ_j^σ , $0 \leq j \leq m-1$, are completely determined by the Frenet formulas, it suffices to prove that the linear frame (X_1, \dots, X_m) satisfies the property (ii) of Proposition 3.2, which is equivalent to prove the existence of functions $h_{ij} \in C^\infty(t_0 - \varepsilon, t_0 + \varepsilon)$, $1 \leq i \leq j \leq m$, such that $\nabla_{T^\sigma}^{j-1} T^\sigma = \sum_{i=1}^j h_{ij} X_i$ for $1 \leq j \leq m$. If $j = 1$, then this formula follows from the first formula in (21) with $h_{11} = \kappa_0$. Hence we can proceed by recurrence on $j \geq 2$. By applying the operator ∇_{T^σ} to both sides of the equation $\nabla_{T^\sigma}^{j-2} T^\sigma = \sum_{i=1}^{j-1} h_{i,j-1} X_i$, we have $\nabla_{T^\sigma}^{j-1} T^\sigma = \sum_{i=1}^{j-1} ((dh_{i,j-1}/dt) X_i + \nabla_{T^\sigma} X_i)$, and the result follows by replacing the term $\nabla_{T^\sigma} X_i = \kappa_0 \nabla_{X_1} X_i$ by its expression deduced from the formulas in (21) above. \square

Theorem 3.7. *Let (M, g) be an m -dimensional oriented Riemannian manifold. Given a system of functions $\kappa = (\kappa_0, \dots, \kappa_{m-1})$, with $\kappa_j \in C^\infty(t_0 - \delta, t_0 + \delta)$ for $0 \leq j \leq m-1$ and $\kappa_j > 0$ for $0 \leq j \leq m-2$, let $f_{ij}^\kappa: (t_0 - \delta, t_0 + \delta) \rightarrow \mathbb{R}$, $1 \leq i \leq j \leq m$, be the functions defined by the following recurrence relations:*

$$(22) \quad \begin{cases} f_{11}^\kappa = \kappa_0, \\ f_{12}^\kappa = \frac{df_{11}^\kappa}{dt}, \\ f_{22}^\kappa = f_{11}^\kappa \kappa_0 \kappa_1, \end{cases}$$

$$(23) \quad 3 \leq j \leq m \quad \left\{ \begin{array}{l} f_{1j}^\kappa = \frac{df_{1,j-1}^\kappa}{dt} - f_{2,j-1}^\kappa \kappa_0 \kappa_1, \\ f_{ij}^\kappa = \frac{df_{i,j-1}^\kappa}{dt} - f_{i+1,j-1}^\kappa \kappa_0 \kappa_i + f_{i-1,j-1}^\kappa \kappa_0 \kappa_{i-1}, \\ 2 \leq i \leq j-2, \\ f_{j-1,j}^\kappa = \frac{df_{j-1,j-1}^\kappa}{dt} + f_{j-2,j-1}^\kappa \kappa_0 \kappa_{j-2}, \\ f_{jj}^\kappa = f_{j-1,j-1}^\kappa \kappa_0 \kappa_{j-1}. \end{array} \right.$$

Let $w_j \in T_{x_0}M$, $1 \leq j \leq m$, be vectors such that the system (w_1, \dots, w_{m-1}) is linearly independent. The necessary and sufficient conditions for a Frenet curve $\sigma: (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow M$, $0 < \varepsilon \leq \delta$, to exist such that,

- (i) $\sigma(t_0) = x_0$,
- (ii) $(\nabla_{T^\sigma}^{j-1} T^\sigma)(t_0) = w_j$ for $1 \leq j \leq m$,
- (iii) $\kappa_{j-1}^\sigma = \kappa_{j-1}$ for $1 \leq j \leq m$,

are the following:

$$(24) \quad g(w_i, w_j) = \sum_{h=1}^i f_{hi}^\kappa(t_0) f_{hj}^\kappa(t_0), \quad 1 \leq i \leq j \leq m.$$

Proof. If the curve σ in the statement exists, then $f_{ij}^\kappa = f_{ij}^\sigma$, where the functions f_{ij}^σ are given in the formulas (8) and (9), and from the formulas (7) we have

$$\begin{aligned} g(w_i, w_j) &= g\left(\nabla_{T^\sigma}^{i-1} T^\sigma, \nabla_{T^\sigma}^{j-1} T^\sigma\right)(t_0) \\ &= \sum_{h=1}^i f_{hi}^\sigma(t_0) f_{hj}^\sigma(t_0). \end{aligned}$$

Hence, all the conditions (24) are necessary for the curve σ to exist.

Let (v_1, \dots, v_{m-1}) be the orthonormal system in $T_{x_0}M$ obtained by applying the Gram-Schmidt process to the system (w_1, \dots, w_{m-1}) , and let v_m be the only unitary tangent vector orthogonal to v_1, \dots, v_{m-1} for which the basis $(v_1, \dots, v_{m-1}, v_m)$ of $T_{x_0}M$ is positively oriented. According to Theorem 3.6, there exists a Frenet curve $\sigma: (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow M$ such that,

- a) $\sigma(t_0) = x_0$; b) $X_j^\sigma(t_0) = v_j$, $1 \leq j \leq m$; c) $\kappa_j^\sigma = \kappa_j$, $0 \leq j \leq m-1$.

Hence $f_{ij}^\kappa = f_{ij}^\sigma$, as follows from the formulas (8), (9), (22), and (23), and from (7) we obtain $(\nabla_{T^\sigma}^{j-1} T^\sigma)(t_0) = \sum_{i=1}^j f_{ij}^\sigma(t_0) v_i$, $1 \leq j \leq m$. Consequently, the Gram-Schmidt process applied to $(T^\sigma(t_0), \dots, (\nabla_{T^\sigma}^{m-2} T^\sigma)(t_0))$ also leads one to the orthonormal system (v_1, \dots, v_{m-1}) . By virtue of (24) we thus have $g(w_i, w_j) = g(\nabla_{T^\sigma}^{i-1} T^\sigma, \nabla_{T^\sigma}^{j-1} T^\sigma)(t_0)$ for $1 \leq i \leq j \leq m$, and we can conclude by simply recalling the following fact: If (u_1, \dots, u_k) , (u'_1, \dots, u'_k) are two linearly independent systems such that, 1st) the Gram-Schmidt process applied to

(u_1, \dots, u_k) , as well as to (u'_1, \dots, u'_k) , leads to the same orthonormal system, and 2nd) $g(u_i, u_j) = g(u'_i, u'_j)$ for $1 \leq i \leq j \leq k$, then both systems coincide, i.e., $u_i = u'_i$ for $i = 1, \dots, k$.

Finally, the Frenet frame at t_0 of any Frenet curve $\sigma: (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow M$ satisfying (i)-(iii) in the statement coincides with the system (v_1, \dots, v_m) and we can conclude its uniqueness from Theorem 3.6. \square

Remark 3.8. The explicit formulas for the functions f_{ij}^κ are rather involved, but their computational evaluation is quite feasible; for example,

For $m = 3$:

$$f_{33}^\kappa = \left(\frac{d^2 \kappa_0}{dt^2} \right)^2 - 2 \frac{d^2 \kappa_0}{dt^2} (\kappa_0)^3 (\kappa_1)^2 + (\kappa_0)^6 (\kappa_1)^4 + 9 \left(\frac{d\kappa_0}{dt} \right)^2 (\kappa_0)^2 (\kappa_1)^2 + 6 \frac{d\kappa_0}{dt} \frac{d\kappa_1}{dt} (\kappa_0)^3 \kappa_1 + \left(\frac{d\kappa_1}{dt} \right)^4 (\kappa_0)^4 + (\kappa_0)^6 (\kappa_1)^2 (\kappa_2)^2.$$

For $m = 4$:

$$f_{24}^\kappa = \frac{d\kappa_0}{dt} \frac{d^3 \kappa_0}{dt^3} - 3 \left(\frac{d\kappa_0}{dt} \right)^2 (\kappa_0)^2 (\kappa_1)^2 + 2 \frac{d\kappa_0}{dt} \frac{d\kappa_1}{dt} (\kappa_0)^3 \kappa_1 + 4 \frac{d^2 \kappa_0}{dt^2} (\kappa_0)^3 (\kappa_1)^2 + \frac{d^2 \kappa_1}{dt^2} (\kappa_0)^4 \kappa_1 - (\kappa_0)^6 (\kappa_1)^2 (\kappa_2)^2 - (\kappa_0)^6 (\kappa_1)^4.$$

Remark 3.9. Although the formulas (10) in the statement of Proposition 3.4 are necessary for the curve σ to exist, they are not sufficient. More formally, let $\Xi_\kappa = (\pi'^m, \Xi_\kappa^1, \dots, \Xi_\kappa^m): J^m(\mathbb{R}, M) \rightarrow M \times \mathbb{R}^m$ be the mapping of fibred manifolds over M given by,

$$\begin{aligned} \Xi_\kappa^1 &= \Delta_1 \circ \pi_1^m - (\kappa_0 \circ \pi^m)^2, \\ \Xi_\kappa^2 &= \Delta_2 \circ \pi_2^m - (\kappa_1 \circ \pi^m)^2 (\Delta_1 \circ \pi_1^m)^3, \\ \Xi_\kappa^i &= (\Delta_{i-2} \circ \pi_{i-2}^m) (\Delta_i \circ \pi_i^m) - (\kappa_{i-1} \circ \pi^m)^2 (\Delta_1 \circ \pi_1^m) (\Delta_{i-1} \circ \pi_{i-1}^m)^2, \\ 3 \leq i &\leq m-1, \\ \Xi_\kappa^m(j_{t_0}^m \sigma) &= \text{vol}_g(T^\sigma, \nabla_{T^\sigma} T^\sigma, \dots, \nabla_{T^\sigma}^{m-1} T^\sigma)(t_0) \sqrt{\Delta_{m-2}^\sigma(t_0)} \\ &\quad - \kappa_{m-1}(t_0) \sqrt{\Delta_1^\sigma(t_0)} \Delta_{m-1}^\sigma(t_0), \end{aligned}$$

$\pi^m: J^m(\mathbb{R}, M) \rightarrow \mathbb{R}$, $\pi'^m: J^m(\mathbb{R}, M) \rightarrow M$ being the canonical projections and $\Delta_k: J^m(\mathbb{R}, M) \rightarrow \mathbb{R}$ being the mapping defined in (11). Because of the expression for Ξ_κ^m , the mapping Ξ_κ is quasi-linear. If $\kappa = \kappa^\sigma$ for a Frenet curve σ , then it is readily checked that the equations $\Xi_{\kappa^\sigma} \circ j^m \sigma = 0$ are equivalent to the formulas (10).

The fibred submanifold $R^m = (\Xi_\kappa)^{-1}\{0\} \cap (\pi_{m-1}^m)^{-1}(\mathcal{F}^{m-1}(M))$ is not formally integrable, even in the simplest cases. For $M = \mathbb{R}^2$, $g = dx^2 + dy^2$, the submanifold R^2 is defined by the equations $\dot{x}^2 + \dot{y}^2 = (\kappa_0)^2$, $\dot{x}\ddot{y} - \ddot{x}\dot{y} = (\kappa_0)^3 \kappa_1$, whereas its first prolongation R^3 is obtained by adding to the equations of R^2 the following: $\dot{x}\ddot{x} + \dot{y}\ddot{y} = \kappa_0 \dot{\kappa}_0$, $\dot{x}\ddot{\ddot{y}} - \ddot{\ddot{x}}\dot{y} = 3(\kappa_0)^2 \dot{\kappa}_0 \kappa_1 + (\kappa_0)^3 \dot{\kappa}_1$. Hence the projection $\pi_2^3: R^3 \rightarrow R^2$ is not surjective. For $M = \mathbb{R}^3$, $g = dx^2 + dy^2 + dz^2$,

the submanifold R^3 is defined by the following equations:

$$\begin{aligned} \dot{x}^2 + \dot{y}^2 + \dot{z}^2 &= (\kappa_0)^2, \\ \begin{vmatrix} \dot{x}^2 + \dot{y}^2 + \dot{z}^2 & \dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} \\ \dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} & \ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2 \end{vmatrix} &= (\kappa_0)^6 (\kappa_1)^2, \\ \begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \ddot{\dot{x}} & \ddot{\dot{y}} & \ddot{\dot{z}} \end{vmatrix} &= (\kappa_0)^6 (\kappa_1)^2 \kappa_2, \end{aligned}$$

and one needs to reach the second prolongation $R^5 \subset J^5(\mathbb{R}, M)$ of R^3 in order to obtain an integrable system. For higher dimensions similar conclusions are obtained.

3.4 $\mathcal{F}^{m-1}(M)$ and $\mathcal{N}^{m-1}(M)$

Theorem 3.10. *Let (M, g) be a Riemannian manifold. Let $\mathcal{F}_{t_0, x_0}^{m-1}(M)$ be the subset of Frenet jets $j_{t_0}^{m-1}\sigma \in J^{m-1}(\mathbb{R}, M)$ such that, $\sigma(t_0) = x_0$. Let $\mathcal{N}_{t_0, x_0}^{m-1}(M)$ be the subset of jets $j_{t_0}^{m-1}\sigma \in J^{m-1}(\mathbb{R}, M)$ such that, i) $\sigma(t_0) = x_0$, ii) the curve σ is normal general position up to the order $m-1$ at t_0 , i.e., the tangent vectors $U_{t_0}^{\sigma, 1}, U_{t_0}^{\sigma, 2}, \dots, U_{t_0}^{\sigma, r}$ defined in (1) are linearly independent.*

If either $\dim M = m \leq 4$ or g is a flat metric at a neighbourhood of x_0 , then $\mathcal{F}_{t_0, x_0}^{m-1}(M) = \mathcal{N}_{t_0, x_0}^{m-1}(M)$, $\forall t_0 \in \mathbb{R}$, $\forall x_0 \in M$.

In the general case, $\mathcal{F}_{t_0, x_0}^{m-1}(M) \setminus \mathcal{N}_{t_0, x_0}^{m-1}(M)$ (resp. $\mathcal{N}_{t_0, x_0}^{m-1}(M) \setminus \mathcal{F}_{t_0, x_0}^{m-1}(M)$) is not empty but nowhere dense in $\mathcal{F}_{t_0, x_0}^{m-1}(M)$ (resp. $\mathcal{N}_{t_0, x_0}^{m-1}(M)$).

Proof. If (U, x^1, \dots, x^m) is the normal coordinate system attached to an orthonormal basis for $T_{x_0}M$ with respect to the Levi-Civita connection ∇ of g , then for every smooth curve $(t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow M$, $\sigma(t_0) = x_0$, the following formulas hold:

$$\begin{aligned} T_{t_0}^\sigma &= U_{t_0}^{\sigma, 1}, \\ (\nabla_{T^\sigma} T^\sigma)_{t_0} &= U_{t_0}^{\sigma, 2}, \\ (\nabla_{T^\sigma}^2 T^\sigma)_{t_0} &= U_{t_0}^{\sigma, 3}, \end{aligned} \tag{25}$$

and for $r \geq 3$, from the formulas (2), (3), (4), (5), we conclude the existence of a polynomial P_i^r in the values $(\partial^{|I|} \Gamma_{jk}^h / \partial x^i)(x_0)$, $I \in \mathbb{N}^m$, $|I| \leq r-2$, Γ_{jk}^h being the Christoffel symbols of ∇ with respect to the coordinates chosen, and the components $(d^k(x^i \circ \sigma)/dt^k)(t_0)$ (also in such coordinates) of the tangent vectors $U_{t_0}^{\sigma, k}$, $1 \leq k \leq r-1$, defined in (1) such that,

$$(\nabla_{T^\sigma}^r T^\sigma)_{t_0} = U_{t_0}^{\sigma, r+1} + P_i^r \frac{\partial}{\partial x^i} \Big|_{x_0}. \tag{26}$$

As the values $(\partial^{|I|} \Gamma_{jk}^h / \partial x^i)(x_0)$, $1 \leq |I| \leq k$, can be written as a polynomial (e.g., see [13], [22]) in the components of the curvature tensor field R^g and its

covariant derivatives $\nabla R^g, \dots, \nabla^{k-1} R^g$ at x_0 , we conclude that the same holds for P_i^r . For example,

$$\begin{aligned} (\nabla_{T^\sigma}^3 T^\sigma)_{t_0} &= U_{t_0}^{\sigma,4} + \frac{1}{3} R_{x_0}^g (T_{t_0}^\sigma, (\nabla_{T^\sigma} T^\sigma)_{t_0}) T_{t_0}^\sigma, \\ (\nabla_{T^\sigma}^4 T^\sigma)_{t_0} &= U_{t_0}^{\sigma,5} + 2 (\nabla R^g)_{x_0} (T_{t_0}^\sigma, (\nabla_{T^\sigma} T^\sigma)_{t_0}, T_{t_0}^\sigma, T_{t_0}^\sigma) \\ &\quad + 3 R_{x_0}^g (T_{t_0}^\sigma, (\nabla_{T^\sigma} T^\sigma)_{t_0}) (\nabla_{T^\sigma} T^\sigma)_{t_0} + \frac{7}{3} R_{x_0}^g (T_{t_0}^\sigma, (\nabla_{T^\sigma}^2 T^\sigma)_{t_0}) T_{t_0}^\sigma. \end{aligned}$$

For $m \leq 4$ from the formulas (25) we conclude

$$(27) \quad \mathcal{F}_{t_0, x_0}^{m-1}(M) = \mathcal{N}_{t_0, x_0}^{m-1}(M).$$

Moreover, the equation $(\nabla_t^r T)_{t_0} = U_{t_0}^{\sigma, r+1}$ holds for $0 \leq r \leq m-2$ if and only if, $P_i^r = 0$ for $0 \leq r \leq m-2$, $1 \leq i \leq m$. In particular, this happens when g is flat at a neighbourhood of x_0 ; hence the equality (27) also holds in this case. If the tangent vectors $T_{t_0}^\sigma, (\nabla_{T^\sigma} T^\sigma)_{t_0}, \dots, (\nabla_{T^\sigma}^{m-2} T^\sigma)_{t_0}$ are linearly independent but there exists a non-trivial linear combination, i.e., $0 = \sum_{h=1}^{m-1} \lambda_h U_{t_0}^{\sigma, h}$, then from (26) we deduce

$$(28) \quad \sum_{r=0}^{m-2} \lambda_{r+1} \left\{ (\nabla_{T^\sigma}^r T^\sigma)_{t_0} - P_i^r (\partial/\partial x^i)_{x_0} \right\} = 0,$$

which implies that at least one of the vectors $P_i^r (\partial/\partial x^i)_{x_0}$, $3 \leq r \leq m-2$, does not vanish.

Letting $N_{t_0} = T_{t_0} \times (\nabla_t T)_{t_0} \times \dots \times (\nabla_t^{m-2} T)_{t_0}$, where \times stands for cross product, we obtain a basis $(T_{t_0}^\sigma, (\nabla_{T^\sigma} T^\sigma)_{t_0}, \dots, (\nabla_{T^\sigma}^{m-2} T^\sigma)_{t_0}, N_{t_0})$ for $T_{x_0} M$, and we can write,

$$P_i^r (\partial/\partial x^i)_{x_0} = \sum_{q=0}^{m-2} \mu_q^r (\nabla_{T^\sigma}^q T^\sigma)_{t_0} + \mu^r N_{t_0}, \quad 0 \leq r \leq m-2,$$

for some scalars μ_q^r, μ^r , agreeing that $P_i^r = 0$ for $0 \leq r \leq 2$; hence $\mu_q^r = \mu^r = 0$ for $0 \leq r \leq 2$. Then, (28) is equivalent to saying that the homogeneous linear system

$$\begin{aligned} \sum_{r=0}^{m-2} \mu^r \lambda_{r+1} &= 0, \\ \sum_{r=0}^{m-2} (\mu_q^r - \delta_q^r) \lambda_{r+1} &= 0, \quad 0 \leq q \leq m-2, \end{aligned}$$

of m equations in the $m-1$ unknowns $\lambda_1, \dots, \lambda_{m-1}$ admits a non-trivial solution, i.e., the rank of the $m \times (m-1)$ matrix

$$\mu(m) = \begin{pmatrix} 0 & 0 & 0 & \mu_0^3 & \dots & \mu^{m-2} \\ -1 & 0 & 0 & \mu_0^3 & \dots & \mu_0^{m-2} \\ 0 & -1 & 0 & \mu_1^3 & \dots & \mu_1^{m-2} \\ 0 & 0 & -1 & \mu_2^3 & \dots & \mu_2^{m-2} \\ 0 & 0 & 0 & \mu_3^3 - 1 & \dots & \mu_3^{m-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \mu_{m-2}^3 & \dots & \mu_{m-2}^{m-2} - 1 \end{pmatrix}$$

must be $\leq m - 2$. This condition characterizes $\mathcal{F}_{t_0, x_0}^{m-1}(M) \setminus \mathcal{N}_{t_0, x_0}^{m-1}(M)$. The proof for $\mathcal{N}_{t_0, x_0}^{m-1}(M) \setminus \mathcal{F}_{t_0, x_0}^{m-1}(M)$ is similar. \square

Example 3.11. For $m = 5$ jets in $\mathcal{F}_{t_0, x_0}^4(M) \setminus \mathcal{N}_{t_0, x_0}^4(M)$ are given by, $\text{rk } \mu(5) = 3$. Hence $\mu^3 = 0$, $\mu_3^3 = 1$; $P_i^3(\partial/\partial x^i)_{x_0} = \sum_{q=0}^2 \mu_q^3 (\nabla_{T^\sigma}^q T^\sigma)_{t_0} + (\nabla_{T^\sigma}^3 T^\sigma)_{t_0}$, i.e.,

$$(29) \quad R_{x_0}^g(T_{t_0}^\sigma, (\nabla_{T^\sigma} T^\sigma)_{t_0}) T_{t_0}^\sigma - (\nabla_{T^\sigma}^3 T^\sigma)_{t_0} \in \langle T_{t_0}^\sigma, (\nabla_{T^\sigma} T^\sigma)_{t_0}, (\nabla_{T^\sigma}^2 T^\sigma)_{t_0} \rangle.$$

In addition, assume $(x^i)_{i=1}^5$ is the normal coordinate system defined by the Frenet frame $(X_i^\sigma(t_0))_{i=1}^5$. From the formulas (7), (8), and (9) it follows the formula (29) can be reformulated by saying that the tangent vector

$$(\kappa_0^\sigma)^4 \kappa_1^\sigma R_{x_0}^g(X_1^\sigma(t_0), X_2^\sigma(t_0)) X_1^\sigma(t_0) - f_{14}^\sigma(t_0) X_1^\sigma(t_0) - f_{24}^\sigma(t_0) X_2^\sigma(t_0) \\ - f_{34}^\sigma(t_0) X_3^\sigma(t_0) - f_{44}^\sigma(t_0) X_4^\sigma(t_0)$$

must belong to $\langle X_1^\sigma(t_0), X_2^\sigma(t_0), X_3^\sigma(t_0) \rangle$, or equivalently,

$$g(R_{x_0}^g(X_1^\sigma(t_0), X_2^\sigma(t_0)) X_1^\sigma(t_0), X_4^\sigma(t_0)) = \kappa_2^\sigma(t_0) \kappa_3^\sigma(t_0), \\ g(R_{x_0}^g(X_1^\sigma(t_0), X_2^\sigma(t_0)) X_1^\sigma(t_0), X_5^\sigma(t_0)) = 0.$$

Example 3.12. For $m=6$ jets in $\mathcal{F}_{t_0, x_0}^5(M) \setminus \mathcal{N}_{t_0, x_0}^5(M)$ are given by, $\text{rk } \mu(6) \leq 4$. The rank of $\mu(6)$ is 3 if and only if, $\mu^3 = \mu^4 = \mu_4^3 = \mu_3^4 = 0$, $\mu_3^3 = \mu_4^4 = 1$, or equivalently,

$$P_i^3(\partial/\partial x^i)_{x_0} = \mu_0^3 T_{t_0}^\sigma + \mu_1^3 (\nabla_{T^\sigma} T^\sigma)_{t_0} + \mu_2^3 (\nabla_{T^\sigma}^2 T^\sigma)_{t_0} + (\nabla_{T^\sigma}^3 T^\sigma)_{t_0}, \\ P_i^4(\partial/\partial x^i)_{x_0} = \mu_0^4 T_{t_0}^\sigma + \mu_1^4 (\nabla_{T^\sigma} T^\sigma)_{t_0} + \mu_2^4 (\nabla_{T^\sigma}^2 T^\sigma)_{t_0} + (\nabla_{T^\sigma}^4 T^\sigma)_{t_0}.$$

In other words, $P_i^3(\partial/\partial x^i)_{x_0} - (\nabla_{T^\sigma}^3 T^\sigma)_{t_0}$ and $P_i^4(\partial/\partial x^i)_{x_0} - (\nabla_{T^\sigma}^4 T^\sigma)_{t_0}$ belong to the subspace spanned by $T_{t_0}^\sigma$, $(\nabla_{T^\sigma} T^\sigma)_{t_0}$, and $(\nabla_{T^\sigma}^2 T^\sigma)_{t_0}$.

The rank of $\mu(6)$ is 4 if and only if,

$$\mu_4^3 \mu_3^4 = (\mu_3^3 - 1)(\mu_4^4 - 1), \\ \mu_4^3 \mu_4^4 = \mu_3^3(\mu_4^4 - 1), \\ \mu_3^4 \mu_3^3 = \mu_4^4(\mu_3^3 - 1),$$

but $(\mu^3, \mu^4, \mu_4^3, \mu_3^4, \mu_3^3, \mu_4^4) \neq (0, 0, 0, 0, 1, 1)$.

4 The equivalence problem

4.1 Necessary conditions for congruence

Definition 4.1. Two curves $\sigma: (a, b) \rightarrow (M, g)$, $\bar{\sigma}: (a, b) \rightarrow (\bar{M}, \bar{g})$ with values in two Riemannian manifolds are said to be *congruent* if an open neighbourhood U of the image of σ in M and an isometric embedding $\phi: U \rightarrow \bar{M}$ exist such that, $\bar{\sigma} = \phi \circ \sigma$. If M and \bar{M} are oriented, then ϕ is assumed to preserve the orientation.

Proposition 4.2. *The curvatures of a Frenet curve with values into an oriented Riemannian manifold (M, g) are invariant by congruence.*

Proof. Let $\sigma: (a, b) \rightarrow M$, $\bar{\sigma}: (a, b) \rightarrow \bar{M}$ be two Frenet curves with values in two oriented Riemannian manifolds (M, g) , (\bar{M}, \bar{g}) . Let $\phi: U \rightarrow \bar{M}$ be an isometric embedding preserving the orientation defined on a neighbourhood U of the image of σ , such that $\bar{\sigma} = \phi \circ \sigma$. Since ϕ is an affine mapping, from [20, VI, Proposition 1.2] we know $\phi \cdot (\nabla_{T^\sigma}^j T^\sigma) = \bar{\nabla}_{T^{\bar{\sigma}}}^j T^{\bar{\sigma}}$, for all $j \in \mathbb{N}$. As ϕ is an isometry, for every $i, j = 1, \dots, m$, the following formula holds:

$$\begin{aligned} \bar{g} \left(\bar{\nabla}_{T^{\bar{\sigma}}}^{i-1} T^{\bar{\sigma}}, \bar{\nabla}_{T^{\bar{\sigma}}}^{j-1} T^{\bar{\sigma}} \right) &= \bar{g} \left(\phi_* \left(\nabla_{T^\sigma}^{i-1} T^\sigma \right), \phi_* \left(\nabla_{T^\sigma}^{j-1} T^\sigma \right) \right) \\ &= g \left(\nabla_{T^\sigma}^{i-1} T^\sigma, \nabla_{T^\sigma}^{j-1} T^\sigma \right). \end{aligned}$$

From Proposition 3.5 it follows $\kappa_i^\sigma = \kappa_i^{\bar{\sigma}}$, $0 \leq i \leq m-2$, $|\kappa_{m-1}^\sigma| = |\kappa_{m-1}^{\bar{\sigma}}|$ and since ϕ is orientation-preserving, we have

$$\text{vol}_g (T^\sigma, \dots, \nabla_{T^\sigma}^{m-1} T^\sigma) = \text{vol}_{\bar{g}} (T^{\bar{\sigma}}, \dots, \bar{\nabla}_{T^{\bar{\sigma}}}^{m-1} T^{\bar{\sigma}}).$$

Thus $\kappa_{m-1}^\sigma = \kappa_{m-1}^{\bar{\sigma}}$, and the proof is complete. \square

Proposition 4.3. *Let (M, g) , (\bar{M}, \bar{g}) be two oriented Riemannian manifolds with associated Levi-Civita connections $\nabla, \bar{\nabla}$, respectively, and let $\sigma: (a, b) \rightarrow M$, $\bar{\sigma}: (a, b) \rightarrow \bar{M}$, be two Frenet curves which are congruent under the isometric embedding ϕ . Then, $\phi \cdot X_i^\sigma = X_i^{\bar{\sigma}}$, $\omega_\sigma^i = \phi^* \omega_{\bar{\sigma}}^i$, for $1 \leq i \leq m$, $(\omega_\sigma^1, \dots, \omega_\sigma^m)$, $(\omega_{\bar{\sigma}}^1, \dots, \omega_{\bar{\sigma}}^m)$ being the dual coframes of the Frenet frames of σ , $\bar{\sigma}$, respectively. Moreover,*

$$\phi_* \left(\nabla^j R \left(X_{i_1}^\sigma, \dots, X_{i_{j+3}}^\sigma, \omega_\sigma^i \right) \right) (\sigma(t)) = \bar{\nabla}^j \bar{R} \left(X_{i_1}^{\bar{\sigma}}, \dots, X_{i_{j+3}}^{\bar{\sigma}}, \omega_{\bar{\sigma}}^i \right) (\bar{\sigma}(t)),$$

for all $j \in \mathbb{N}$, $t \in (a, b)$, and all systems of indices $i, i_1, \dots, i_{j+3} = 1, \dots, m$, where R, \bar{R} are the curvature tensors of (M, g) , (\bar{M}, \bar{g}) , respectively.

Proof. From Proposition 4.2 and the formulas (8), (9), we obtain $f_{ij}^\sigma = f_{ij}^{\bar{\sigma}}$ for every $i, j = 1, \dots, m$. Therefore

$$\begin{aligned} \sum_{i=1}^m f_{im}^\sigma X_i^\sigma &= \sum_{i=1}^m f_{im}^{\bar{\sigma}} X_i^{\bar{\sigma}} \\ &= \bar{\nabla}_{T^{\bar{\sigma}}}^{m-1} T^{\bar{\sigma}} \\ &= \phi_* \left(\nabla_{T^\sigma}^{m-1} T^\sigma \right) \\ &= \phi_* \left(\sum_{i=1}^m f_{im}^\sigma X_i^\sigma \right) \\ &= \sum_{i=1}^m f_{im}^\sigma (\phi_* X_i^\sigma). \end{aligned}$$

Thus $\bar{X}_i^{\bar{\sigma}} = \phi_* X_i^\sigma$ (and therefore $\omega_\sigma^i = \phi^* \omega_{\bar{\sigma}}^i$), $1 \leq i \leq m$. This proves the first part of the statement. The second one derives from [20, VI, Proposition 1.2]. \square

4.2 General criterion of congruence

Theorem 4.4. *Let (M, g) , (\bar{M}, \bar{g}) be two oriented connected Riemannian manifolds of class C^ω of the same dimension, $m = \dim M = \dim \bar{M}$, with Levi-Civita connections ∇ , $\bar{\nabla}$, and let $\sigma: (a, b) \rightarrow M$, $\bar{\sigma}: (a, b) \rightarrow \bar{M}$ be two Frenet curves of class C^ω with tangent fields T , \bar{T} , respectively. If $x_0 = \sigma(t_0)$, $\bar{x}_0 = \bar{\sigma}(t_0)$, $a < t_0 < b$, then σ and $\bar{\sigma}$ are congruent on some neighbourhoods of x_0 and \bar{x}_0 , respectively, if and only if the following conditions hold:*

- (i) *For every $j \in \mathbb{N}$ and every $0 \leq i \leq m-1$,*

$$(30) \quad \frac{d^j \kappa_i^\sigma}{dt^j}(t_0) = \frac{d^j \kappa_i^{\bar{\sigma}}}{dt^j}(t_0),$$

- (ii) *For every $j \in \mathbb{N}$ and all systems of indices $i, i_1, \dots, i_{j+3} = 1, \dots, m$, the following formula holds:*

$$(31) \quad (\nabla^j R) \left(X_{i_1}^\sigma, \dots, X_{i_{j+3}}^\sigma, \omega_\sigma^i \right) (x_0) = (\bar{\nabla}^j \bar{R}) \left(X_{i_1}^{\bar{\sigma}}, \dots, X_{i_{j+3}}^{\bar{\sigma}}, \omega_{\bar{\sigma}}^i \right) (\bar{x}_0),$$

where $(\omega_\sigma^1, \dots, \omega_\sigma^m)$, $(\omega_{\bar{\sigma}}^1, \dots, \omega_{\bar{\sigma}}^m)$ are the dual coframes of the Frenet frames $(X_1^\sigma, \dots, X_m^\sigma)$, $(X_1^{\bar{\sigma}}, \dots, X_m^{\bar{\sigma}})$ of σ , $\bar{\sigma}$, and R , \bar{R} are the curvature tensors of (M, g) , (\bar{M}, \bar{g}) , respectively.

Proof. From Proposition 4.2 (resp. Proposition 4.3) the equations (30) (resp. (31)) follow. To prove the converse, let $A: T_{x_0} M \rightarrow T_{\bar{x}_0} \bar{M}$ be the linear isometry given by, $A(X_i^\sigma(t_0)) = X_i^{\bar{\sigma}}(t_0)$, $1 \leq i \leq m$. The condition (31) implies that A maps the tensor $(\nabla^j R)_{x_0}$ into the tensor $(\bar{\nabla}^j \bar{R})_{\bar{x}_0}$, for all $j \in \mathbb{N}$. From [20, VI, Theorem 7.2] we conclude that the polar map $\phi: U \rightarrow \bar{U}$, $\phi = \exp_{\bar{x}_0} \circ A \circ \exp_{x_0}^{-1}$, is an affine isomorphism and from [36, Lemma 2.3.1] it follows that ϕ is an isometry. In order to finish the proof, it suffices to check that $\phi(\sigma(t)) = \bar{\sigma}(t)$ for $|t - t_0| < \varepsilon$, and a small enough $\varepsilon > 0$. The Frenet curve $\gamma = \phi \circ \sigma: (a, b) \rightarrow \bar{M}$ satisfies $\gamma(t_0) = \bar{x}_0$, $X_i^\gamma(t_0) = \phi_*(X_i^\sigma(t_0)) = X_i^{\bar{\sigma}}(t_0)$. As the curvatures are of class C^ω , from the condition (30) we deduce $\kappa_j^{\bar{\sigma}} = \kappa_j^\sigma$, $0 \leq j \leq m-1$, and since the curvatures are invariant by congruence, we know $\kappa_j^{\bar{\sigma}} = \kappa_j^\gamma$; hence $\kappa_j^{\bar{\sigma}} = \kappa_j^\gamma$, $0 \leq j \leq m-1$. Taking the formulas (7), (9) and the condition (30) into account for $1 \leq j \leq m$ we have

$$\begin{aligned} A \left(\left(\nabla_{T^\sigma}^{j-1} T^\sigma \right)_{t_0} \right) &= \sum_{i=1}^m f_{ij}^\sigma(t_0) A(X_i^\sigma(t_0)) \\ &= \sum_{i=1}^m f_{ij}^{\bar{\sigma}}(t_0) X_i^{\bar{\sigma}}(t_0) \\ &= \left(\bar{\nabla}_{T^{\bar{\sigma}}}^{j-1} T^{\bar{\sigma}} \right)_{t_0}. \end{aligned}$$

Therefore

$$\begin{aligned}
\left(\bar{\nabla}_{T^\gamma}^{j-1} T^\gamma\right)_{t_0} &= \phi_* \left(\left(\nabla_{T^\sigma}^{j-1} T^\sigma\right)_{t_0} \right) \\
&= A \left(\left(\nabla_{T^\sigma}^{j-1} T^\sigma\right)_{t_0} \right) \\
&= \left(\bar{\nabla}_{T^{\bar{\sigma}}}^{j-1} T^{\bar{\sigma}}\right)_{t_0},
\end{aligned}$$

for $1 \leq j \leq m$. By applying Theorem 3.6 we conclude $\bar{\sigma} = \gamma = \phi \circ \sigma$ on $(t_0 - \varepsilon, t_0 + \varepsilon)$. \square

Corollary 4.5. *Let (M, g) , (\bar{M}, \bar{g}) be two oriented connected Riemannian manifolds of class C^ω of the same dimension, $m = \dim M = \dim \bar{M}$, and let $\sigma: (a, b) \rightarrow M$, $\bar{\sigma}: (a, b) \rightarrow \bar{M}$ be two Frenet curves, respectively. If $x_0 = \sigma(t_0)$, $\bar{x}_0 = \bar{\sigma}(t_0)$, $a < t_0 < b$, then σ and $\bar{\sigma}$ are congruent on some neighbourhoods U and \bar{U} of x_0 and \bar{x}_0 , respectively if, and only if, the following conditions hold:*

- (i) *For every $j \in \mathbb{N}$ and every $0 \leq i \leq m-1$, it holds $\kappa_i^\sigma(t) = \kappa_i^{\bar{\sigma}}(t)$, for $|t - t_0| < \varepsilon$.*
- (ii) *For every $j \in \mathbb{N}$ and every system of indices $i, i_1, \dots, i_{j+3} \in \{1, \dots, m\}$,*

$$(\nabla^j R) \left(X_{i_1}^\sigma, \dots, X_{i_{j+3}}^\sigma, \omega_\sigma^i \right) (x_0) = (\bar{\nabla}^j \bar{R}) \left(X_{i_1}^{\bar{\sigma}}, \dots, X_{i_{j+3}}^{\bar{\sigma}}, \omega_{\bar{\sigma}}^i \right) (\bar{x}_0).$$

4.3 Remarks on the criterion of congruence

Remark 4.6. The condition (31) of Theorem 4.4 is *not* equivalent to the following:

$$(32) \quad R(X_j^\sigma, X_k^\sigma, X_l^\sigma, \omega_\sigma^i)(\sigma(t)) = \bar{R}(X_j^{\bar{\sigma}}, X_k^{\bar{\sigma}}, X_l^{\bar{\sigma}}, \omega_{\bar{\sigma}}^i)(\bar{\sigma}(t)), \quad |t - t_0| < \varepsilon.$$

Differentiating the left-hand side of (32) we have

$$\begin{aligned}
\frac{d}{dt} R(X_j^\sigma, X_k^\sigma, X_l^\sigma, \omega_\sigma^i)(\sigma(t)) &= (\nabla_{T^\sigma} R)(X_j^\sigma, X_k^\sigma, X_l^\sigma, \omega_\sigma^i)(\sigma(t)) \\
&\quad + R(\nabla_{T^\sigma} X_j^\sigma, X_k^\sigma, X_l^\sigma, \omega_\sigma^i)(\sigma(t)) + \dots \\
&\quad + R(X_j^\sigma, X_k^\sigma, X_l^\sigma, \nabla_{T^\sigma} \omega_\sigma^i)(\sigma(t)) \\
&= \kappa_0^\sigma(t) \nabla R(X_1^\sigma, X_j^\sigma, X_k^\sigma, X_l^\sigma, \omega_\sigma^i)(\sigma(t)) + \dots
\end{aligned}$$

As the first argument of ∇R in the formula above is X_1^σ , the function

$$\nabla R(X_h^\sigma, X_j^\sigma, X_k^\sigma, X_l^\sigma, \omega_\sigma^i), \quad h \neq 1,$$

cannot be recovered from $R(X_j^\sigma, X_k^\sigma, X_l^\sigma, \omega_\sigma^i)(\sigma(t))$. Therefore the formulas (32) do not imply the formulas (31), although (31) do imply (32) as the manifolds involved are analytic.

Example 4.7. Let us consider the bidimensional torus $\mathbb{T} \subset \mathbb{R}^3$ with implicit equation $(x^2 + y^2 + z^2 + 3)^2 = 16(x^2 + y^2)$. On the radius-2 circumference $C = \mathbb{T} \cap \{z = 1\}$ the Gaussian curvature of \mathbb{T} vanishes and C is a regular curve of positive constant curvature. The curvature tensor of \mathbb{R}^2 vanishes in particular along any curve $C' \subset \mathbb{R}^2$ with the same curvature as C ; but C and C' are not congruent since the Gaussian curvature of \mathbb{T} does not vanish at every neighbourhood of a point of C .

Remark 4.8. From Proposition 3.2 we deduce that the Frenet frame of a Frenet curve σ and its dual frame at a point $\sigma(t_0)$ depend on $j_{t_0}^{m-1}\sigma$ only. Hence, for every system of indices $j \in \mathbb{N}$, $i_1, \dots, i_{j+3}, i \in \{1, \dots, m\}$, a function $I_{i_1 \dots i_{j+3}, i}^j: \mathcal{F}^{m-1}(M) \rightarrow \mathbb{R}$ can be defined on the open subset $\mathcal{F}^{m-1}(M) \subset J^{m-1}(\mathbb{R}, M)$ of the jets of order $m-1$ of Frenet curves with values in M by setting

$$(33) \quad I_{i_1 \dots i_{j+3}, i}^j(j_t^{m-1}\sigma) = (\nabla^j R) \left(X_{i_1}^\sigma, \dots, X_{i_{j+3}}^\sigma, \omega_\sigma^i \right) (\sigma(t)).$$

Similarly, for every $0 \leq i \leq m-1$, a function

$$(34) \quad \varkappa_i: (\pi_{m-1}^m)^{-1} \mathcal{F}^{m-1}(M) \subset J^m(\mathbb{R}, M) \rightarrow \mathbb{R}$$

can be defined by setting $\varkappa_i(j_t^m \sigma) = \kappa_i^\sigma(t)$.

From Theorem 4.4 it follows that all these functions are invariant by isometry (see the section 5 below). Since $\dim \mathcal{F}^{m-1}(M) = m^2 + 1$, only a finite number (not greater than $m^2 + 1$) of such functions can be functionally independent generically. Hence, the infinite number of conditions given in (31) can be reduced to a finite number. Nevertheless, it is not easy to determine a bound for the index j , which measures the times one has to differentiate covariantly the curvature tensor. In Theorem 8.10 below this bound is proved to be 2 in the case of a surface.

Remark 4.9. For the sake of simplicity, here we use the Riemann curvature tensor R_4 of g rather than the curvature tensor R (cf. [20, V, Section 2]), i.e.,

$$R_4(X, Y, T, Z) = g(R(T, Z)Y, X).$$

For $j = 0$, all the functions

$$\begin{aligned} I_{i_1 i_2 i_3 i_4}: \mathcal{F}^{m-1}(M) &\rightarrow \mathbb{R}, \\ I_{i_1 i_2 i_3 i_4}(j_t^{m-1}\sigma) &= R_4(X_{i_1}^\sigma(t), X_{i_2}^\sigma(t), X_{i_3}^\sigma(t), X_{i_4}^\sigma(t)) \end{aligned}$$

can be written in terms of the functions

$$\begin{aligned} I_{ij}: FM &\rightarrow \mathbb{R}, \\ I_{ij}(X_1, \dots, X_m) &= R_4(X_i, X_j, X_i, X_j), \\ X_i, X_j &\in T_x M, \quad 1 \leq i < j \leq m, \end{aligned}$$

where FM is the bundle of linear frames of M , as follows from the polarization formula, namely

$$\begin{aligned}
6R(X, Y, T, Z) = & R(X, Z, X, Z) + R(T, Y, T, Y) - R(X, T, X, T) - R(Z, Y, Z, Y) \\
& - R(X, Y + Z, X, Y + Z) + R(X, Y + T, X, Y + T) \\
& - R(T, Y + Z, T, Y + Z) + R(Z, Y + T, Z, Y + T) \\
& + R(X + T, Y + Z, X + T, Y + Z) + R(X + Z, T, X + Z, T) \\
& - R(X + T, Y, X + T, Y) - R(X + T, Z, X + T, Z) \\
& - R(X + Z, T + Y, X + Z, T + Y) + R(X + Z, Y, X + Z, Y).
\end{aligned}$$

In fact, if $\mathbf{f}_M: \mathcal{F}^{m-1}(M) \rightarrow FM$, $s_{ij}: FM \rightarrow FM$ are the maps

$$\begin{aligned}
\mathbf{f}_M(j_t^{m-1}\sigma) &= (X_1^\sigma(t), \dots, X_m^\sigma(t)), \\
s_{ij}(X_1, \dots, X_m) &= (X_1, \dots, X_i, \dots, X_i + X_j, \dots, X_m), \quad i < j,
\end{aligned}$$

then

$$\begin{aligned}
6I_{i_1 i_2 i_3 i_4} = & (I_{i_1 i_4} + I_{i_2 i_3} - I_{i_1 i_3} - I_{i_2 i_4} - I_{i_1 i_4} \circ s_{i_2 i_4} + I_{i_1 i_3} \circ s_{i_2 i_3} - I_{i_3 i_4} \circ s_{i_2 i_4} \\
& + I_{i_3 i_4} \circ s_{i_2 i_3} + I_{i_3 i_4} \circ s_{i_2 i_4} \circ s_{i_1 i_3} + I_{i_3 i_4} \circ s_{i_1 i_4} - I_{i_2 i_3} \circ s_{i_1 i_3} \\
& - I_{i_3 i_4} \circ s_{i_1 i_3} - I_{i_3 i_4} \circ s_{i_2 i_3} \circ s_{i_1 i_4} + I_{i_2 i_4} \circ s_{i_1 i_4}) \circ \mathbf{f}_M.
\end{aligned}$$

Remark 4.10. Theorem 4.4 is the most general result we can expect without imposing any additional condition on (M, g) and (\bar{M}, \bar{g}) except for the fact of being analytic. This is principally due to the fact that [20, VI, Theorem 7.2] cannot be generalized to non-analytic manifolds, as shown in the next example.

Example 4.11. Let g, \bar{g} be the two Riemannian metrics on $M = \bar{M} = \mathbb{R}^m$, $m \geq 2$, defined by,

$$\begin{aligned}
g_{ij}(x) &= \delta_{ij} + \exp(-|x|^{-2}), \\
\bar{g}_{ij}(x) &= \delta_{ij},
\end{aligned}$$

respectively; hence (M, g) is not analytic at the origin. If R is the curvature tensor of (M, g) and ∇ is its associated Levi-Civita connection, then $(\nabla^n R)(0) = 0$ for all $n \in \mathbb{N}$. The identity map $Id: T_0 M \rightarrow T_0 \bar{M}$ is an isometry, since $g_{ij}(0) = \bar{g}_{ij}(0) = \delta_{ij}$. Moreover, $(\nabla^j R)(0) = (\bar{\nabla}^j \bar{R})(0) = 0$, where \bar{R} (resp. $\bar{\nabla}$) is the curvature tensor (resp. the Levi-Civita connection) of \bar{g} . If there exists an affine isomorphism $\phi: U \rightarrow \bar{U} = \bar{M}$, defined on normal neighbourhoods of 0, such that $\phi_{*,0} = Id$, then taking [36, Lemma 2.3.1] into account, ϕ must necessarily be an isometry. Hence ϕ maps the tensor $\nabla^j R$ into the tensor $\bar{\nabla}^j \bar{R} = 0$, for all $j \in \mathbb{N}$. Consequently, $\nabla^j R$ must vanish in a normal neighbourhood of 0, but this is not true. In fact, as $g_{ij} = \delta_{ij} + h(|x|)$, with $h(s) = \exp(-s^{-2})$, we have

$$g^{ij} = \delta_{ij} - \frac{h(|x|)}{1 + mh(|x|)}.$$

Following the notation in [20], the Christoffel symbols are,

$$\Gamma_{ij}^k = \frac{h'(|x|)}{|x|} \left(x^i + x^j - x^k + \frac{h(|x|)}{1 + mh(|x|)} \left(\sum_{a=1}^m x^a - m(x^i + x^j) \right) \right).$$

If $x_t = (t, \dots, t) \in \mathbb{R}^m$, $t \neq 0$, then

$$\Gamma_{ij}^k(x_t) = \frac{h'(|x_t|)t}{|x_t|(1 + mh(|x_t|))} \neq 0,$$

$$\begin{aligned} \Gamma_{ii}^a(x_t) \Gamma_{aj}^j(x_t) &= \left(\frac{h'(|x_t|)}{|x_t|(1 + mh(|x_t|))} \right)^2 t^2 \\ &= \Gamma_{ji}^a(x_t) \Gamma_{ai}^j(x_t). \end{aligned}$$

Consequently, $\Gamma_{ii}^a(x_t) \Gamma_{aj}^j(x_t) - \Gamma_{ji}^a(x_t) \Gamma_{ai}^j(x_t) = 0$, and hence

$$\begin{aligned} R_{iji}^j(x_t) &= \frac{\partial \Gamma_{ii}^j}{\partial x^j}(x_t) - \frac{\partial \Gamma_{ji}^j}{\partial x^i}(x_t) \\ &= \frac{-2h'(|x_t|)}{|x_t|} \\ &\neq 0, \end{aligned}$$

Thus, R_{iji}^j does not vanish at x_t for small enough $t \neq 0$.

5 Differential invariants

5.1 Basic definitions

Let $\mathfrak{I}(M, g)$ be the group of isometries of a complete Riemannian connected manifold (M, g) endowed with its structure of Lie transformation group (cf. [20, VI, Theorem 3.4]) and let $\mathfrak{i}(M, g)$ be its Lie algebra, which is anti-isomorphic to the algebra of Killing vector fields.

Every diffeomorphism $\phi: M \rightarrow M$ induces a transformation $\phi^{(r)}$ on $J^r(\mathbb{R}, M)$ given by $\phi^{(r)}(j_t^r \sigma) = j_t^r(\phi \circ \sigma)$, and a natural action (on the left) of the group $\mathfrak{I}(M, g)$ on $J^r(\mathbb{R}, M)$ can be defined by $\phi \cdot j_t^r \sigma = \phi^{(r)}(j_t^r \sigma)$. Each $X \in \mathfrak{i}(M, g)$ induces a flow ϕ_t and its jet prolongation $\phi_t^{(r)}$ determines a flow on $J^r(\mathbb{R}, M)$, the infinitesimal generator of which is the vector field denoted by $X^{(r)} \in \mathfrak{X}(J^r(\mathbb{R}, M))$. The tangent spaces to the orbits of the action of $\mathfrak{I}(M, g)$ on $J^r(\mathbb{R}, M)$ coincide with the fibres of the distribution $\mathfrak{D}^r \subset \mathfrak{X}(J^r(\mathbb{R}, M))$ spanned by the vector fields $X^{(r)}$; more precisely, we have

$$T_{j_t^r \sigma}(\mathfrak{I}(M, g) \cdot j_t^r \sigma) = \mathfrak{D}_{j_t^r \sigma}^r = \left\{ X_{j_t^r \sigma}^{(r)} : X \in \mathfrak{i}(M, g) \right\}.$$

Definition 5.1. A smooth function $I: J^r(\mathbb{R}, M) \rightarrow \mathbb{R}$ is said to be an *invariant* of order r (cf. [1, 7, 4.1], [23]) if, $I \circ \phi^{(r)} = I$, $\forall \phi \in \mathfrak{J}(M, g)$. A first integral $f: J^r(\mathbb{R}, M) \rightarrow \mathbb{R}$ of the distribution \mathfrak{D}^r is called a *differential invariant* of order r ; i.e., $X^{(r)}(f) = 0$, $\forall X \in \mathfrak{i}(M, g)$.

Remark 5.2. A differential invariant is an invariant with respect to the connected component of the identity $\mathfrak{J}^0(M, g)$ in $\mathfrak{J}(M, g)$.

Lemma 5.3 (cf. [29]). *The distribution \mathfrak{D}^r is involutive and its rank is locally constant on a dense open subset $\mathcal{U}^r \subseteq J^r(\mathbb{R}, M)$. If N_r denotes the maximal number of functionally independent differential invariants of order $r \geq 0$, then*

$$(35) \quad \begin{aligned} N_r &= \dim J^r(\mathbb{R}, M) - \text{rk } \mathfrak{D}^r|_{\mathcal{U}^r} \\ &= m(r+1) + 1 - \text{rk } \mathfrak{D}^r|_{\mathcal{U}^r}. \end{aligned}$$

Proof. \mathfrak{D}^r is involutive as $[X_1^{(r)}, X_2^{(r)}] = [X_1, X_2]^{(r)}$, $\forall X_1, X_2 \in \mathfrak{i}(M, g)$. Let \mathcal{U}^r be the subset defined as follows: A point $\xi = j_t^r \sigma \in J^r(\mathbb{R}, M)$ belongs to \mathcal{U}^r if and only if ξ admits an open neighbourhood N_ξ such that $\dim \mathfrak{D}_{\xi'}^r = \dim \mathfrak{D}_\xi^r$ for every $\xi' \in N_\xi$. As $N_\xi \subseteq \mathcal{U}^r$, it follows that \mathcal{U}^r is an open subset, which is non-empty as the dimension of the fibres of \mathfrak{D}^r is uniformly bounded and hence, \mathcal{U}^r contains the points ξ for which $\dim \mathfrak{D}_\xi^r = \max_{\xi' \in J^r(\mathbb{R}, M)} \dim \mathfrak{D}_{\xi'}^r = d$. In fact, if this equation holds, then there exists an open neighbourhood N_ξ of ξ such that the dimension of the fibres of \mathfrak{D}^r over the points $\xi' \in N_\xi$ is at least d , as if $(X_i^{(r)})_\xi$, $1 \leq i \leq d$, is a basis for \mathfrak{D}_ξ^r , then the vector fields $(X_i^{(r)})$ are linearly independent at each point of an open neighbourhood and hence, they are also a basis, d being the maximal value of the dimension of the fibres of \mathfrak{D}^r . From the very definition of \mathcal{U}^r we thus conclude that $N_\xi \subseteq \mathcal{U}^r$. The same argument proves that the rank of \mathfrak{D}^r is locally constant over \mathcal{U}^r . Next, we prove that \mathcal{U}^r is dense. If $O \subset J^r(\mathbb{R}, M)$ is a non-empty open subset, then there exists $\xi \in O$ such that $\dim \mathfrak{D}_\xi^r = \max_{\xi' \in O} \dim \mathfrak{D}_{\xi'}^r$, and we can conclude as above. The last part of the statement follows directly from the Frobenius theorem. \square

If f is a differential invariant of order r , then $D_t(f)$ is a differential invariant of order $r+1$. This fact follows from the formula $X^{(r+1)} \circ D_t = D_t \circ X^{(r)}$ for every $X \in \mathfrak{i}(M, g)$, which, in its turn, follows from the formula

$$(36) \quad X^{(r)} = \sum_{j=0}^r (D_t)^j (f^j) \frac{\partial}{\partial x_j^i},$$

for every $X \in \mathfrak{X}(M)$ with local expression

$$(37) \quad X = f^i \frac{\partial}{\partial x^i}, \quad f^i \in C^\infty(M).$$

If $\pi_l^k: J^k(\mathbb{R}, M) \rightarrow J^l(\mathbb{R}, M)$ is the canonical projection for $k > l$, then

$$(\pi_{r-1}^r)_* X^{(r)} = X^{(r-1)}, \quad \forall X \in \mathfrak{X}(M),$$

and the following exact sequence defines the subdistribution $\mathfrak{D}^{r, r-1}$:

$$(38) \quad 0 \rightarrow \mathfrak{D}_{j_t^r \sigma}^{r, r-1} \rightarrow \mathfrak{D}_{j_t^r \sigma}^r \xrightarrow{(\pi_{r-1}^r)^*} \mathfrak{D}_{j_t^{r-1} \sigma}^{r-1} \rightarrow 0, \quad \forall j_t^r \sigma \in J^r(\mathbb{R}, M).$$

5.2 Stability

Theorem 5.4. *Let (M, g) be a complete Riemannian connected manifold and let $\sigma: (a, b) \rightarrow M$ be a smooth curve such that $j_{t_0}^{m-1}\sigma \in \mathcal{N}^{m-1}(M)$, $a \leq t_0 \leq b$, with the same notations as in section 3.4. If $X \in \mathfrak{i}(M, g)$ is a Killing vector field such that $X_{j_{t_0}^{m-1}\sigma}^{(m-1)} = 0$, $m = \dim M$, then $X = 0$.*

Proof. Let $x_0 = \sigma(t_0)$ and let $U \subset T_{x_0}(M)$ be an open neighbourhood of the origin on which the exponential mapping $\exp: T_{x_0}(M) \rightarrow M$ is a diffeomorphism onto its image. Let $(X_j)_{j=1}^m$ be a g -orthonormal basis for $T_{x_0}M$ with dual basis $(w^i)_{i=1}^m$, $w^i \in T_{x_0}^*(M)$ (i.e., $w^i(X_j) = \delta_j^i$) and let $x^i = w^i \circ (\exp|_U)^{-1}$, $1 \leq i \leq m$, be the corresponding normal coordinate system.

If $\phi: M \rightarrow M$ is an affine transformation of the Levi-Civita connection of g (in particular, if ϕ is an isometry of g) leaving the point x_0 invariant, then (cf. [20, VI, Proposition 1.1]), $\phi \circ \exp = \exp \circ \phi_*$, $\phi_*: T_{x_0}(M) \rightarrow T_{x_0}(M)$ being the Jacobian mapping at x_0 . Hence $x^i \circ \phi = (w^i \circ \phi_*) \circ (\exp|_U)^{-1}$. If $\phi_*(X_j) = a_j^i X_i$, then $w^i \circ \phi_* = a_h^i w^h$ and hence, $x^i \circ \phi = a_h^i x^h$. In particular, if ϕ_τ is the flow of a Killing vector field X locally given as in (37), then

$$\begin{aligned} x^i \circ \phi_\tau &= a_h^i(\tau) x^h, \quad (a_h^i(\tau))_{h,i=1}^m \in O(m), \quad \forall \tau \in \mathbb{R}, \\ f^i &= b_h^i x^h, \quad b_h^i = \frac{da_h^i}{d\tau}(0), \quad B = (b_h^i)_{h,i=1}^m \in \mathfrak{so}(m). \end{aligned}$$

According to (36), the assumption on X in the statement is equivalent to saying

$$\frac{d^k(f^i \circ \sigma)}{dt^k}(t_0) = 0, \quad 1 \leq i \leq m, \quad 0 \leq k \leq m-1,$$

or equivalently, $b_h^i(d^k(x^h \circ \sigma)/dt^k)(t_0) = 0$, i.e., $B(U_{t_0}^{\sigma,k}) = 0$, $1 \leq k \leq m-1$, the tangent vectors $U_{t_0}^{\sigma,k}$ being defined in the formula (1). As B is skew-symmetric, we also have $B(U_{t_0}^{\sigma,1} \times \dots \times U_{t_0}^{\sigma,m-1}) = 0$. Therefore, $B = 0$. \square

Corollary 5.5. *On a complete Riemannian connected manifold (M, g) the order of asymptotic stability (cf. [2], [23]) of the algebra of differential invariants, is $\leq m$. Accordingly, $N_r = (r+1)m + 1 - \dim \mathfrak{i}(M, g)$, $\forall r \geq m-1$.*

Proof. This follows from the previous theorem and the exact sequence (38), taking account of the fact that $X_{j_t^{m-1}\sigma}^{(m-1)} = 0$ if and only if $X_{j_t^m\sigma}^{(m)} \in \mathfrak{D}_{j_t^m\sigma}^{m,m-1}$. \square

Corollary 5.6. *On a complete Riemannian connected manifold (M, g) the distribution \mathfrak{D}^{m-1} takes its maximal rank on $\mathcal{N}^{m-1}(M)$.*

Proof. If $j_t^{m-1}\sigma \in \mathcal{N}^{m-1}(M)$, then from Theorem 5.4 it follows that the linear map $\mathfrak{i}(M, g) \rightarrow \mathfrak{D}_{j_t^{m-1}\sigma}^{m-1}$, $X \mapsto X_{j_t^{m-1}\sigma}^{(m-1)}$, is an isomorphism. \square

5.3 Completeness

Let a Lie group G act on a manifold N . If the quotient manifold $q: N \rightarrow N/G$ exists, then the image of the mapping $q^*: C^\infty(N/G) \rightarrow C^\infty(N)$, $f \mapsto f \circ q$, is the subalgebra of G -invariant functions, namely $C^\infty(N)^G = q^*C^\infty(N/G)$. Below we are concerned with the case $G = \mathcal{I}^0(M, g)$ acting on $N = J^r(\mathbb{R}, M)$ as defined at the beginning of the section 5.1.

Definition 5.7. Let $O^r \subseteq J^r(\mathbb{R}, M)$ be an invariant open subset under the natural action of the group $\mathcal{I}^0(M, g)$. A system of invariant functions $I_i: O^r \rightarrow \mathbb{R}$, $1 \leq i \leq \nu$, is said to be *complete* if the equations $I_i(j_{t_0}^r \sigma) = I_i(j_{t_0}^r \sigma')$, $1 \leq i \leq \nu$, $j_{t_0}^r \sigma, j_{t_0}^r \sigma' \in O^r$, imply that σ and σ' are congruent on a neighbourhood of t_0 .

Proposition 5.8. Let ∇ be a linear connection on M . The mapping Φ_∇^r defined in the formula (6) makes the diagram

$$\begin{array}{ccc} J^r(\mathbb{R}, M) & \xrightarrow{\Phi_\nabla^r} & \mathbb{R} \times \oplus^r TM \\ \phi^{(r)} \downarrow & & \downarrow (1_{\mathbb{R}} \oplus \phi_*^r) \\ J^r(\mathbb{R}, M) & \xrightarrow{\Phi_{\phi \cdot \nabla}^r} & \mathbb{R} \times \oplus^r TM \end{array}$$

commutative for every $\phi \in \text{Diff}(M)$.

Proof. The proof is a consequence of the formula (2) and Lemma 3.6, taking the definition of $\phi \cdot \nabla$ into account. \square

Theorem 5.9. Let (M, g) be a complete oriented connected Riemannian manifold of class C^ω . If

$$(39) \quad I_i: (\pi_{m-1}^r)^{-1} \mathcal{F}^{m-1}(M) \rightarrow \mathbb{R}, \quad r \geq m, \quad 1 \leq i \leq \nu,$$

is a complete system of invariants, then there exists a dense open subset O^r in $(\pi_{m-1}^r)^{-1} \mathcal{F}^{m-1}(M)$ such that $I_i|_{O^r}$, $1 \leq i \leq \nu$, generate the ring of differential invariants under the group $\mathcal{I}^0(M, g)$ on an open neighbourhood $N^r \subseteq O^r$ of every point $j_t^r \sigma \in O^r$, i.e.,

$$C^\infty(N^r)^{\mathcal{I}^0(M, g)} = (I_1|_{N^r}, \dots, I_\nu|_{N^r})^* C^\infty(\mathbb{R}^\nu).$$

Conversely, if a system of functions as in (39) locally generates the ring of invariants over a dense subset $\tilde{O}^r \subseteq (\pi_{m-1}^r)^{-1} \mathcal{F}^{m-1}(M)$, then it is complete.

Proof. According to Theorem 5.4, we can confine ourselves to prove the statement for $r = m$. First of all, we prove that the quotient manifold

$$q^m: (\pi_{m-1}^m)^{-1} \mathcal{F}^{m-1}(M) \rightarrow (\pi_{m-1}^m)^{-1} \mathcal{F}^{m-1}(M) / \mathcal{I}^0(M, g) = Q^m$$

exists. To this end, by applying [24, Theorem 9.16], we only need to prove that the following two conditions hold:

1. The isotropy subgroup $\mathcal{I}^0(M, g)_{j_t^m \sigma}$ reduces to the identity map of M for every $j_t^m \sigma$ in $(\pi_{m-1}^m)^{-1} \mathcal{F}^{m-1}(M)$.

2. $\mathcal{I}^0(M, g)$ acts properly on $(\pi_{m-1}^m)^{-1}\mathcal{F}^{m-1}(M)$.

The image of $(\pi_{m-1}^m)^{-1}(\mathcal{F}^{m-1})$ by the diffeomorphism Φ_{∇}^m is equal to the subset $U^m \subset \mathbb{R} \times \oplus^m TM$ of elements (t, X_1, \dots, X_m) such that (X_1, \dots, X_{m-1}) are linearly independent tangent vectors. From Proposition 5.8 we deduce that an isometry ϕ belongs to the isotropy subgroup $\mathcal{I}^0(M, g)_{j_t^m \sigma}$ of a point $j_t^m \sigma$ in $(\pi_{m-1}^m)^{-1}\mathcal{F}^{m-1}(M)$, if and only $(1_{\mathbb{R}}, \oplus^m \phi_*(\sigma(t)))$ belongs to the isotropy subgroup of the point $\Phi_{\nabla}^m(j_t^m \sigma) = (t, X_1, \dots, X_m) \in U^m$. Hence $\phi = Id_M$ and consequently, $\mathcal{I}^0(M, g)$ acts freely on $(\pi_{m-1}^m)^{-1}(\mathcal{F}^{m-1})$, thus proving the first item above.

Moreover, if g_1 is the Sasakian metric induced by g on TM (e.g., see [4, 1.K], [16, Section 7], [38, IV, Section 1]), then $\mathcal{I}^0(M, g)$ acts by isometries of the metric on $J^m(\mathbb{R}, M)$ given by

$$\mathbf{g}^m = (\Phi_{\nabla}^m)^* \left(dt^2 + \sum_{i=1}^m (\text{pr}_i)^* g_1 \right),$$

where $\text{pr}_i: \mathbb{R} \times \oplus^m TM \rightarrow TM$ is the projection $\text{pr}_i(t, X_1, \dots, X_m) = X_i$, and the image of the mapping $\mathcal{I}^0(M, g) \rightarrow \mathcal{I}^0(J^m(\mathbb{R}, M), \mathbf{g}^m)$, $\phi \mapsto \phi^{(m)}$, is closed as it is defined by the following closed conditions:

$$\varphi^*(t) = t, (j^m \sigma)^*(\varphi^* \omega) = 0, \quad \varphi \in \mathcal{I}^0(J^m(\mathbb{R}, M), \mathbf{g}^m),$$

for every $\sigma \in C^\infty(\mathbb{R}, M)$ and every contact 1-form ω on $J^m(\mathbb{R}, M)$, by virtue of [37, Theorem 3.1]. From [31, 5.2.4. Proposition] we conclude the second item above.

The invariant functions $I_i: (\pi_{m-1}^m)^{-1}\mathcal{F}^{m-1}(M) \rightarrow \mathbb{R}$, $1 \leq i \leq \nu$, induce smooth functions on the quotient manifold, $\bar{I}_i: Q^m \rightarrow \mathbb{R}$. As I_1, \dots, I_ν is a complete system of invariants, the mapping $\Upsilon: Q^m \rightarrow \mathbb{R}^\nu$ whose components are $\bar{I}_1, \dots, \bar{I}_\nu$, is injective.

The same argument as in the proof of Lemma 5.3 states the following property: If $\phi: N \rightarrow N'$ is a smooth mapping, then the subset of the points $x \in N$ for which there exists an open neighbourhood $U(x) \subseteq N$ such that $\phi|_{U(x)}$ is a mapping of constant rank, is a dense open subset in N . Hence an injective smooth map $\phi: N \rightarrow N'$ is an immersion on a dense open subset in N (cf. [24, Theorem 7.15-(b)]). By applying this result to Υ , we conclude the existence of a dense open subset $\bar{O}^m \subseteq Q^m$ such that $\Upsilon|_{\bar{O}^m}$ is an injective immersion. Hence for every $q^m(j_t^m \sigma) \in \bar{O}^m$ there exists a system of coordinates on Q^m defined on an open neighbourhood of $q^m(j_t^m \sigma)$ constituted by some functions $\bar{I}_{i_1}, \dots, \bar{I}_{i_k}$, $k = \dim Q^m$. As $(q^m)^* C^\infty(Q^m)$ can be identified to the ring of differential invariants, we can take $O^m = (q^m)^{-1}(\bar{O}^m)$.

Conversely, if $j_{t_0}^m \sigma, j_{t_0}^m \sigma' \in \bar{O}^m$ are such that $q^m(j_{t_0}^m \sigma') \neq q^m(j_{t_0}^m \sigma)$, then there exists $\rho \in C^\infty(q^m \bar{O}^m)$ satisfying $\rho(q^m(j_{t_0}^m \sigma)) = 0$, $\rho(q^m(j_{t_0}^m \sigma')) = 1$. As $\rho \circ q^m$ is an invariant function on \bar{O}^m by virtue of the hypothesis there exists $f \in C^\infty(\mathbb{R}^\nu)$ such that $\rho \circ (q^m) = f \circ (I_1|_{\bar{O}^m}, \dots, I_\nu|_{\bar{O}^m})$. Hence an index i must exist for which $I_i(j_{t_0}^m \sigma) \neq I_i(j_{t_0}^m \sigma')$, thus proving that I_1, \dots, I_ν is a complete system of invariants. \square

Remark 5.10. If the injective immersion $\Upsilon: \bar{O}^m \rightarrow \mathbb{R}^\nu$ is a closed map, then one has

$$C^\infty(O^m)^{\mathcal{I}^0(M,g)} = (I_1, \dots, I_\nu)^* C^\infty(\mathbb{R}^\nu).$$

5.4 Generating complete systems of invariants

Theorem 5.11. *For every $r \in \mathbb{N}$, let k_r be the maximal number of generically functionally independent r -th order invariants not belonging to the closed—in the C^∞ topology—subalgebra generated by the invariants of order $< r$, and their derivatives with respect to the operator D_t . Then*

$$(40) \quad k_r = N_r - 1 - \sum_{i=0}^{r-1} (r+1-i)k_i,$$

$$(41) \quad m = \sum_{i=0}^m k_i.$$

Hence for every complete Riemannian manifold of dimension m , there exist m generically independent invariants generating a complete system of invariant functions by adding their derivatives with respect to D_t for every order $r \leq m$. Moreover, $k_r = 0$, $\forall r > m$.

Proof. Let $t, I_i^0 \in C^\infty(M)$, $1 \leq i \leq k_0 \leq m$, $N_0 = 1 + k_0$, be a maximal system of invariant functions of order zero. (If (M, g) is not homogeneous, then there exist zero-order differential invariants independent of t ; because of this the proof must start on this order.) Therefore, the rank of the Jacobian matrix $\mathcal{J}^0(I_1^0, \dots, I_{k_0}^0) = (\partial I_i^0 / \partial x^j)_{1 \leq i \leq k_0, 1 \leq j \leq m}$ must be maximal; namely, $\text{rk } \mathcal{J}^0(I_1^0, \dots, I_{k_0}^0) = k_0$. Moreover, one has

$$\frac{\partial(D_t f)}{\partial x_{r+1}^i} = \frac{\partial f}{\partial x_r^i}, \quad \forall f \in C^\infty(J^r(\mathbb{R}, M)),$$

as $[\partial / \partial x_{r+1}^i, D_t] = \partial / \partial x_r^i$, $\forall r \in \mathbb{N}$. Hence the Jacobian matrix of the functions $I_1^0, \dots, I_{k_0}^0, D_t I_1^0, \dots, D_t I_{k_0}^0$ on $J^1(\mathbb{R}, M)$ is of the form

$$\mathcal{J}^1(I_1^0, \dots, I_{k_0}^0, D_t I_1^0, \dots, D_t I_{k_0}^0) = \begin{pmatrix} \left(\frac{\partial I_i^0}{\partial x^j} \right) & 0 \\ \star & \left(\frac{\partial I_i^0}{\partial x^j} \right) \end{pmatrix},$$

and $\text{rk } \mathcal{J}^1(I_1^0, \dots, I_{k_0}^0, D_t I_1^0, \dots, D_t I_{k_0}^0) = 2k_0 \leq N_1 - 1$. We can thus complete the previous system with $k_1 = N_1 - 1 - 2k_0$ new functionally independent invariants $I_1^1, \dots, I_{k_1}^1$, in such a way that the Jacobian matrix of the full system is as follows:

$$\mathcal{J}^1(I_1^0, \dots, I_{k_0}^0, D_t I_1^0, \dots, D_t I_{k_0}^0, I_1^1, \dots, I_{k_1}^1) = \begin{pmatrix} \left(\frac{\partial I_i^0}{\partial x^j} \right) & 0 \\ \star & \left(\frac{\partial I_i^0}{\partial x^j} \right) \\ \star & \left(\frac{\partial I_i^1}{\partial x_j^1} \right) \end{pmatrix},$$

with $\text{rk } \mathcal{J}^1(I_1^0, \dots, I_{k_0}^0, D_t I_1^0, \dots, D_t I_{k_0}^0, I_1^1, \dots, I_{k_1}^1) = 2k_0 + k_1 = N_1 - 1$. Let us consider the second-order invariants

$$\begin{aligned} & I_1^0, \dots, I_{k_0}^0, \\ & D_t I_1^0, \dots, D_t I_{k_0}^0, I_1^1, \dots, I_{k_1}^1, \\ & D_t^2 I_1^0, \dots, D_t^2 I_{k_0}^0, D_t I_1^1, \dots, D_t I_{k_1}^1, \end{aligned}$$

the Jacobian matrix of which is

$$\begin{pmatrix} \left(\frac{\partial I_i^0}{\partial x^j} \right) & 0 & 0 \\ \star & \left(\frac{\partial I_i^0}{\partial x^j} \right) & 0 \\ \star & \left(\frac{\partial I_i^1}{\partial x_1^j} \right) & 0 \\ \star & \star & \left(\frac{\partial I_i^0}{\partial x^j} \right) \\ \star & \star & \left(\frac{\partial I_i^1}{\partial x_1^j} \right) \end{pmatrix}$$

and its rank is equal to $3k_0 + 2k_1$. Hence we need to choose k_2 new second-order invariants $I_1^2, \dots, I_{k_2}^2$, with $k_2 = N_2 - 1 - (3k_0 + 2k_1)$, such that the matrix

$$\begin{pmatrix} \left(\frac{\partial I_i^0}{\partial x^j} \right) & 0 & 0 \\ \star & \left(\frac{\partial I_i^0}{\partial x^j} \right) & 0 \\ \star & \left(\frac{\partial I_i^1}{\partial x_1^j} \right) & 0 \\ \star & \star & \left(\frac{\partial I_i^0}{\partial x^j} \right) \\ \star & \star & \left(\frac{\partial I_i^1}{\partial x_1^j} \right) \\ \star & \star & \left(\frac{\partial I_i^2}{\partial x_2^j} \right) \end{pmatrix}$$

is of maximal rank, i.e., $3k_0 + 2k_1 + k_2 = N_2 - 1$. Proceeding step by step in the same way, we conclude that the formula (40) in the statement holds true. Moreover, from this formula we obtain $N_r - N_{r-1} = \sum_{i=0}^r k_i$. According to Corollary 5.5, we have $N_r = (r+1)m + 1 - \dim \mathfrak{i}(M, g)$, $\forall r \geq m-1$, and then $\sum_{i=0}^m k_i = N_m - N_{m-1} = m$, thus proving the formula (41) in the statement and finishing the proof. \square

Remark 5.12. Following the same notations as in [14], let $H_{r+1} = \mathcal{I}^0(M, g)_{j_t^r \sigma}$ be the isotropy subgroup of a point $j_t^r \sigma$ belonging to the open subset \mathcal{U}^r where the distribution \mathfrak{D}^r is of constant rank (see Lemma 5.3). In [14] the following formula is mentioned: $k_r = \dim H_{r-1} + \dim H_{r+1} - 2 \dim H_r$ in the homogeneous case, i.e., $M = G/H$. Nevertheless, this formula holds on an arbitrary Riemannian manifold and it is an easy consequence of the formulas (35) and (40). In fact, one has, $\dim H_{r+1} = \dim \mathcal{I}^0(M, g) - \text{rk } \mathfrak{D}^r|_{\mathcal{U}^r}$. Hence

$$\dim H_{r-1} + \dim H_{r+1} - 2 \dim H_r = N_{r-2} + N_r - 2N_{r-1} = k_r.$$

6 Congruence on symmetric manifolds

Theorem 6.1. *Let (M, g) , (\bar{M}, \bar{g}) be two locally symmetric Riemannian manifolds of the same dimension, $m = \dim M = \dim \bar{M}$, and let $\sigma: (a, b) \rightarrow M$, $\bar{\sigma}: (a, b) \rightarrow \bar{M}$ be two Frenet curves. If $x_0 = \sigma(t_0)$, $\bar{x}_0 = \bar{\sigma}(t_0)$, $a < t_0 < b$, then σ and $\bar{\sigma}$ are congruent on some neighbourhoods U and \bar{U} of x_0 and \bar{x}_0 , respectively if, and only if, the following conditions hold:*

1. For every $j \in \mathbb{N}$ and every $0 \leq i \leq m-1$,

$$(42) \quad \kappa_i^\sigma(t) = \kappa_i^{\bar{\sigma}}(t), \quad |t - t_0| < \varepsilon.$$

2. For every $i, j, k, l = 1, \dots, m$,

$$(43) \quad R(X_i^\sigma, X_j^\sigma, X_k^\sigma, \omega_\sigma^l)(x_0) = \bar{R}(X_i^{\bar{\sigma}}, X_j^{\bar{\sigma}}, X_k^{\bar{\sigma}}, \omega_{\bar{\sigma}}^l)(\bar{x}_0),$$

$(X_1^\sigma, \dots, X_m^\sigma)$, $(X_1^{\bar{\sigma}}, \dots, X_m^{\bar{\sigma}})$ being the Frenet frames of σ , $\bar{\sigma}$, with dual coframes $(\omega_\sigma^1, \dots, \omega_\sigma^m)$, $(\omega_{\bar{\sigma}}^1, \dots, \omega_{\bar{\sigma}}^m)$, and R, \bar{R} the curvature tensors of (M, g) , (\bar{M}, \bar{g}) , respectively.

Proof. From [20, VI, Theorem 7.7] we know every locally symmetric Riemannian manifold is analytic, and its associated Levi-Civita connection ∇ also is, so we can apply Corollary 4.5. In this case, the conditions (31) are simply (43). \square

Theorem 6.2. *Let (M, g) be an arbitrary Riemannian manifold verifying the following property: Two Frenet curves $\sigma, \bar{\sigma}: (a, b) \rightarrow M$, $\sigma(t_0) = \bar{\sigma}(t_0) = x_0$, are congruent on some neighbourhood of x_0 (preserving the orientation if $\dim M$ is even and reversing the orientation if $\dim M$ is odd) if and only if the conditions (42) and (43) of Theorem 6.1 hold. Then, (M, g) is locally symmetric.*

Proof. Let us fix an orientation on a neighbourhood of $x_0 \in M$, and let $(v_i)_{i=1}^m$ be a positive orthonormal basis of $T_{x_0}M$. Let $\kappa_j \in C^\infty(t_0 - \delta, t_0 + \delta)$, $0 \leq j \leq m-1$, $\delta > 0$, be functions such that $\kappa_j > 0$ for $0 \leq j \leq m-2$. From Theorem 3.6 we know there exist two Frenet curves $\sigma, \bar{\sigma}: (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow M$, $0 < \varepsilon < \delta$, such that $\sigma(t_0) = \bar{\sigma}(t_0) = x_0$ and $X_i^\sigma(t_0) = -X_i^{\bar{\sigma}}(t_0) = v_i$ (hence $\omega_\sigma^i(t_0) = -\omega_{\bar{\sigma}}^i(t_0)$) for $1 \leq i \leq m$, with the same curvatures κ_j , $0 \leq j \leq m-1$. If $\dim M$ is odd, we considered the opposite orientation to construct $\bar{\sigma}$. According to this choice of orientations constructing the Frenet curves σ and $\bar{\sigma}$, for every $i, j, k, l = 1, \dots, m$, we have

$$\begin{aligned} R(X_i^\sigma, X_j^\sigma, X_k^\sigma, \omega_\sigma^l)(x_0) &= R(-X_i^{\bar{\sigma}}, -X_j^{\bar{\sigma}}, -X_k^{\bar{\sigma}}, -\omega_{\bar{\sigma}}^l)(x_0) \\ &= R(X_i^{\bar{\sigma}}, X_j^{\bar{\sigma}}, X_k^{\bar{\sigma}}, \omega_{\bar{\sigma}}^l)(x_0). \end{aligned}$$

From the hypothesis, an open neighbourhood U of the image of σ and an isometric embedding $\phi: U \rightarrow M$ (leaving x_0 fixed) exist such that $\bar{\sigma} = \phi \circ \sigma$. Moreover $\phi_*(X_i^\sigma(t_0)) = X_i^{\bar{\sigma}}(t_0) = -X_i^\sigma(t_0)$, $1 \leq i \leq m$. Thus $\phi_* = -Id_{T_{x_0}M}$. Since $x_0 \in M$ is arbitrary, we can conclude. \square

Remark 6.3. Let (M, g) be a Riemannian symmetric space. Let $G = \mathcal{I}^0(M, g)$ be the connected component of the identity in the group of isometries and let H be the isotropy subgroup of the point x_0 . As usual, we set $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and \mathfrak{m} is identified to $T_{x_0}M$. According to [20, XI, Theorem 3.2], the following formula holds: $R(X, Y)Z = -[[X, Y], Z]$ for every $X, Y, Z \in \mathfrak{m}$. Therefore, if a basis (Y_1, \dots, Y_μ) for \mathfrak{h} is fixed, then

$$\begin{aligned} [X_i^\sigma, X_j^\sigma] &= c_{ij}^\alpha Y_\alpha, & c_{ij}^\alpha &\in \mathbb{R}, \\ [Y_\alpha, X_k^\sigma] &= d_{\alpha k}^h X_h^\sigma, & d_{\alpha k}^h &\in \mathbb{R}. \end{aligned}$$

Hence, $g([X_i^\sigma, X_j^\sigma], X_k^\sigma) = c_{ij}^\alpha d_{\alpha k}^l$. Consequently, the condition (43) can be rewritten as $c_{ij}^\alpha d_{\alpha k}^l = \bar{c}_{ij}^\alpha \bar{d}_{\alpha k}^l$, with the obvious notations for the curve $\bar{\sigma}$.

Remark 6.4. According to Theorem 6.1, on a locally symmetric Riemannian manifold (M, g) the $\frac{1}{2}m(m+1)$ functions $I_{ij} \circ \mathbf{f}_M$, $1 \leq i < j \leq m$, and \varkappa_k , $0 \leq k \leq m-1$, defined in the formulas (33) and (34) of Remark 4.8, respectively, constitute a complete system of differential invariants in the sense that two curves with values in (M, g) are congruent if and only if the functions $I_{ij} \circ \mathbf{f}_M$, \varkappa_k take the same values on both curves.

Example 6.5. For $M = \mathbb{C}P^n$, from the formula for the curvature tensor in [20, XI, p. 277], the Riemann curvature reads

$$\begin{aligned} R_4(X_i, X_j, X_k, X_l) &= \frac{c}{4} \{ \delta_{jl} \delta_{ik} - \delta_{jk} \delta_{il} + g(X_j, JX_l) g(JX_k, X_i) \\ &\quad - g(X_j, JX_k) g(JX_l, X_i) + 2g(X_k, JX_l) g(JX_j, X_i) \}, \end{aligned}$$

J being the canonical complex structure. In particular,

$$R_4(X_i, X_j, X_i, X_j) = \frac{c}{4} (1 - \delta_{ij} + 3\omega(X_i, X_j)^2),$$

where ω is the canonical Kähler 2-form in $\mathbb{C}P^n$. Therefore, if we define the functions $\varpi_{ij}: (\pi_{m-1}^m)^{-1} \mathcal{F}^{m-1}(M) \subset J^m(\mathbb{R}, M) \rightarrow \mathbb{R}$, $1 \leq i < j \leq m$, as

$$\varpi_{ij}(j_{t_0}^m \sigma) = \omega(X_i^\sigma(t_0), X_j^\sigma(t_0)),$$

the family $\{\varpi_{ij}\}_{i < j}$ together with $\{\varkappa_i\}_{i=0}^{m-1}$ constitute a complete system of differential invariants.

7 Congruence on constant curvature manifolds

Theorem 7.1. *Two Frenet curves $\sigma, \bar{\sigma}: (a, b) \rightarrow (M, g)$ taking values in an oriented Riemannian manifold of constant curvature are congruent on some neighbourhoods U and \bar{U} of $x_0 = \sigma(t_0)$ and $\bar{x}_0 = \bar{\sigma}(t_0)$, $a < t_0 < b$, respectively if and only, $\kappa_i^\sigma(t) = \kappa_i^{\bar{\sigma}}(t)$ for $0 \leq i \leq m-1$ and small enough $|t - t_0|$. Conversely, if on an oriented Riemannian manifold (M, g) two arbitrary Frenet curves $\sigma, \bar{\sigma}: (a, b) \rightarrow (M, g)$ are congruent on some neighbourhoods of $x_0 = \sigma(t_0)$, $\bar{x}_0 = \bar{\sigma}(t_0)$, $a < t_0 < b$, if and only if $\kappa_i^\sigma(t) = \kappa_i^{\bar{\sigma}}(t)$ for $0 \leq i \leq m-1$ and small enough $|t - t_0|$, then (M, g) is a manifold of constant curvature.*

Proof. On a manifold of constant curvature k one has ([20, V. Corollary 2.3]): $R(X, Y)Z = k(g(Y, Z)X - g(X, Z)Y)$; hence

$$R(X_i^\sigma, X_j^\sigma, X_k^\sigma, \omega_\sigma^l) = k(\delta_{jk}\delta_i^l - \delta_{ik}\delta_j^l).$$

It follows that on a manifold of constant curvature the equation (43) holds identically. The first part of the statement thus follows from Theorem 6.1.

Let $(v_1, \dots, v_m), (w_1, \dots, w_m)$ be two positively-oriented orthonormal bases in $T_{x_0}M$.

From Theorem 3.6 there exist two Frenet curves $\sigma, \bar{\sigma}: (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow M$, $\varepsilon > 0$, such that,

- i) $\sigma(t_0) = \bar{\sigma}(t_0) = x_0$,
- ii) $X_i^\sigma(t_0) = v_i, X_i^{\bar{\sigma}}(t_0) = w_i, 1 \leq i \leq m$,
- iii) $\kappa_i^\sigma(t) = \kappa_i^{\bar{\sigma}}(t), 0 \leq i \leq m-1, |t - t_0| < \varepsilon$.

By virtue of the hypothesis, there exists an isometry ϕ defined on a neighbourhood of x_0 , fixing x_0 , such that $\phi \circ \sigma = \bar{\sigma}$. Hence $\phi_*(v_i) = w_i, 1 \leq i \leq m$, and accordingly M is an isotropic manifold (e.g., see [34]) and therefore of constant curvature. \square

Two r -jets of curves $j_{t_0}^r \sigma, j_{t_0}^r \bar{\sigma} \in J_{t_0}^r(\mathbb{R}, M)$ are said to be r -congruent, which is denoted by $j_{t_0}^r \sigma \sim_r j_{t_0}^r \bar{\sigma}$, if there exists an isometry ϕ defined on a neighbourhood of $\sigma(t_0)$ such that $\phi^{(r)}(j_{t_0}^r \sigma) = j_{t_0}^r \bar{\sigma}$. It is obvious that “to be r -congruent” is an equivalence relation on $J_{t_0}^r(\mathbb{R}, M)$.

Theorem 7.2. *For every $1 \leq r \leq m$, let $x_{hi} = x_{ih}, h, i = 1, \dots, r$, be the standard coordinate system in $S^2(\mathbb{R}^r)$. Given an oriented Riemannian manifold (M, g) of constant curvature, the mapping $f_M^r: J_{t_0}^r(\mathbb{R}, M) \rightarrow S^2(\mathbb{R}^r)$ with components $(x_{hi} \circ f_M^r)(j_{t_0}^r \sigma) = g(\nabla_{T^\sigma}^{h-1} T^\sigma, \nabla_{T^\sigma}^{i-1} T^\sigma)(t_0), h, i = 1, \dots, r$, induces a homeomorphism between $J_{t_0}^r(\mathbb{R}, M)/\sim_r$ and the submanifold with corners $Q^r \subset S^2(\mathbb{R}^r)$ of the positive semidefinite symmetric matrices. Hence, if M is simply connected and complete, then $J_{t_0}^r(\mathbb{R}, M)/\mathfrak{I}(M, g) \cong Q^r$.*

Proof. A symmetric matrix $A \in S^2(\mathbb{R}^r)$ belongs to Q^r if and only if all its principal minors are nonnegative, thus proving that Q^r is a submanifold with corners. Certainly, the following properties hold: i) $f_M^r(J_{t_0}^r(\mathbb{R}, M)) \subset Q^r$, and ii) $f_M^r(j_{t_0}^r \sigma) = f_M^r(j_{t_0}^r \bar{\sigma})$ if $j_{t_0}^r \sigma$ and $j_{t_0}^r \bar{\sigma}$ are r -congruent.

Firstly, we prove that $f_M^r: J_{t_0}^r(\mathbb{R}, M) \rightarrow Q^r, r \leq m$, is surjective. Given $A = (a_{hi})_{h,i=1}^r \in Q^r$, let $A' = (a'_{hi})_{h,i=1}^m \in Q^m$ be the matrix obtained by letting $a'_{hi} = a_{hi}$ for $i \leq r$, and $a'_{hi} = \delta_{hi}$ for $r+1 \leq i \leq m$. If there exists $j_{t_0}^m \sigma$ such that $f_M^m(j_{t_0}^m \sigma) = A'$, then $f_M^r(j_{t_0}^r \sigma) = A$. Therefore, we can assume $r = m$. In this case, as $A \in Q^m$, there exists a $m \times m$ matrix $B = (b_{hi})_{h,i=1}^m$ such that $A = BB^t$. Let (v_1, \dots, v_m) be a positively oriented orthonormal basis in $T_{x_0}M$. It suffices to prove the existence of a curve $\sigma: (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow M$, such that, $\sigma(t_0) = x_0$ and $(\nabla_{T^\sigma}^{h-1} T^\sigma)_{t_0} = \sum_{i=1}^m b_{hi} v_i, 1 \leq h \leq m$. This fact is a consequence of the existence and uniqueness theorem for ordinary differential systems, taking the formulas (2) into account.

If $f_M^r(j_{t_0}^r \sigma) = f_M^r(j_{t_0}^r \tau)$, then the Gram matrices of the systems $(\nabla_{T^\sigma}^{h-1} T^\sigma)_{t_0}$, $(\nabla_{T^\tau}^{h-1} T^\tau)_{t_0}$, $1 \leq h \leq r$, are equal and consequently, there exists a linear isometry $L: (T_{x_0} M, g_{x_0}) \rightarrow (T_{x_0} M, g_{x_0})$ such that, $L(\nabla_{T^\sigma}^{h-1} T^\sigma)_{t_0} = (\nabla_{T^\tau}^{h-1} T^\tau)_{t_0}$ for $1 \leq h \leq r$. As (M, g) is of constant curvature, there exists a local isometric embedding ϕ of (M, g) such that, $\phi(x_0) = x_0$, $\phi_*(x_0) = L$, and one thus deduces $\phi^{(r)}(j_{t_0}^r \sigma) = j_{t_0}^r \tau$, i.e., $j_{t_0}^r \sigma \sim_r j_{t_0}^r \tau$. Hence, f_M^r induces a continuous bijection $f_M^r: J_{t_0}^r(\mathbb{R}, M)/\sim_r \rightarrow Q^r$, which is a homeomorphism as f_M^r is a proper map. \square

Remark 7.3. The points in the interior of Q^r correspond with r -jets of curves such that $(T_{t_0}, (\nabla_t T)_{t_0}, \dots, (\nabla_t^{r-1} T)_{t_0})$ are linearly independent. In particular, $Q^{m-1} \setminus \partial Q^{m-1}$ correspond with the $(m-1)$ -jets of Frenet curves.

Corollary 7.4. *The ring of differential invariants of order $r \leq m$ over a manifold of constant curvature is isomorphic to $C^\infty(Q^r)$.*

8 Some examples

8.1 Euclidean space

Theorem 8.1. *If $g = (dx^1)^2 + \dots + (dx^m)^2$ is the Euclidean metric on \mathbb{R}^m and $j_{t_0}^{m-1} \sigma \in J^{m-1}(\mathbb{R}, \mathbb{R}^m)$ is a Frenet jet, then*

$$\dim \mathfrak{D}_{j_{t_0}^r \sigma}^r = \begin{cases} m + \frac{1}{2}(2m - r - 1)r, & 1 \leq r \leq m - 1, \\ m + \binom{m}{2}, & r \geq m. \end{cases}$$

Hence

$$N_r = \begin{cases} 1 + \binom{r+1}{2}, & 1 \leq r \leq m - 1, \\ 1 + \frac{1}{2}m(2r - m + 1), & r \geq m. \end{cases}$$

Moreover, the functions t and $D_t^h \varkappa_i: (\pi_{m-1}^m)^{-1} \mathcal{F}^{m-1}(M) \subset J^m(\mathbb{R}, M) \rightarrow \mathbb{R}$, $h + i \leq m - 1$, where $\varkappa_0, \dots, \varkappa_{m-1}$ are defined in the formula (34), are a basis for the algebra of differential invariants of order $\leq m$.

Proof. The Lie algebra $\mathfrak{i}(\mathbb{R}^m, g)$ has the basis constituted by the m translations $T_h = \partial/\partial x^h$, $1 \leq h \leq m$, and the $\frac{1}{2}m(m-1)$ rotations $R_{ij} = x^j \partial/\partial x^i - x^i \partial/\partial x^j$, $1 \leq i < j \leq m$. As T_1, \dots, T_m span \mathfrak{D}^0 over $C^\infty(\mathbb{R}^m)$, one has $\dim \mathfrak{D}_{(t,x)}^0 = m$, $\forall (t, x) \in J^0(\mathbb{R}, M) = \mathbb{R} \times M$. Let $(x^i)_{i=1}^m$ be the only Euclidean coordinates centred at $x_0 = \sigma(t_0)$ such that $X_i^\sigma(t_0) = (\partial/\partial x^i)_{x_0}$, $1 \leq i \leq m$, where $(X_i^\sigma)_{i=1}^m$ denotes the Frenet frame for σ . Let $\text{prol}_{j_{t_0}^r \sigma}^r: \mathfrak{i}(\mathbb{R}^m, g) \rightarrow \mathfrak{D}_{j_{t_0}^r \sigma}^r$ be the mapping $\text{prol}_{j_{t_0}^r \sigma}^r(X) = X_{j_{t_0}^r \sigma}^{(r)}$. According to the formula (36) for the jet prolongation of a vector field, a translation T belongs to $\ker \text{prol}_{j_{t_0}^r \sigma}^r$ if and only if $T = 0$, and the rotations $R = a_j^i x^j \partial/\partial x^i$, $A = (a_j^i)_{i,j=1}^m \in \mathfrak{so}(m)$, belonging to $\ker \text{prol}_{j_{t_0}^r \sigma}^r$ are characterized by

$$\sum_{k=0}^r \sum_{i,j=1}^m a_j^i x_k^j (j_{t_0}^r \sigma) \left. \frac{\partial}{\partial x_k^i} \right|_{j_{t_0}^r \sigma} = 0,$$

or equivalently, $A \cdot U_{t_0}^{\sigma,k} = 0$, $1 \leq k \leq r$, with the same notations as in the formula (1). As the metric is flat, one has $U^{\sigma,k} = \nabla_{T^\sigma}^{k-1} T^\sigma$ for every $k \geq 1$, and we conclude

$$\begin{aligned} \langle U_{t_0}^{\sigma,1}, \dots, U_{t_0}^{\sigma,r} \rangle &= \langle X_1^\sigma(t_0), \dots, X_r^\sigma(t_0) \rangle \\ &= \langle \partial/\partial x^1|_{x_0}, \dots, \partial/\partial x^r|_{x_0} \rangle. \end{aligned}$$

Hence $R \in \ker \text{prol}_{j_{t_0}^r \sigma}^r$ if and only the kernel of the matrix A contains the subspace $\mathbb{R}^r \subset \mathbb{R}^m$ of m -uples whose last $m - r$ components vanish; it thus follows that $\ker \text{prol}_{j_{t_0}^r \sigma}^{(r)}$ is generated by R_{ij} for $r + 1 \leq i < j \leq m$, when $r < m - 1$; for $r \geq m - 1$, we have $\ker \text{prol}_{j_{t_0}^r \sigma}^r = \{0\}$. Hence

$$\begin{aligned} \dim \mathfrak{D}_{j_{t_0}^r \sigma}^r &= \binom{m}{2} - \binom{m-r}{2} \\ &= \frac{1}{2}(2m - r - 1)r. \end{aligned}$$

Moreover, if $\sum_{0 \leq h+i \leq m-1} \lambda_h^i d(D_t^h \mathfrak{x}_i)_{j_{t_0}^m \sigma} = 0$, then by applying this equation to $\partial/\partial x_m^j|_{j_{t_0}^m \sigma}$ and taking the formula $[\partial/\partial x_r^j, D_t] = \partial/\partial x_{r-1}^j$ into account, one obtains

$$\begin{aligned} 0 &= \sum_{i=0}^{m-1} \lambda_{m-1-i}^i \frac{\partial}{\partial x_m^j} (D_t^{m-1-i} \mathfrak{x}_i) (j_{t_0}^m \sigma) \\ &= \lambda_0^{m-1} \frac{\partial \mathfrak{x}_{m-1}}{\partial x_m^j} (j_{t_0}^m \sigma) + \lambda_1^{m-2} \frac{\partial \mathfrak{x}_{m-2}}{\partial x_{m-1}^j} (j_{t_0}^m \sigma) + \dots + \lambda_{m-1}^0 \frac{\partial \mathfrak{x}_0}{\partial x_1^j} (j_{t_0}^m \sigma) \\ &= \sum_{i=1}^m k_j^i \lambda_{m-i}^{i-1}, \quad 1 \leq j \leq m, \end{aligned}$$

where

$$k_j^i = \frac{\partial \mathfrak{x}_{i-1}}{\partial x_i^j} (j_{t_0}^m \sigma), \quad i, j = 1, \dots, m.$$

By proceeding by recurrence on $h + i$, it suffices to prove $\lambda_{m-i}^{i-1} = 0$, $1 \leq i \leq m$.

Let $\pi^r: J^r(\mathbb{R}, M) \rightarrow \mathbb{R}$, $\pi'^r: J^r(\mathbb{R}, M) \rightarrow M$ be the projections $\pi^r(j_t^r \sigma) = t$, $\pi'^r(j_t^r \sigma) = \sigma(t)$, and let \mathcal{U}^k be the local section of the vector bundle $(\pi'^m)^* TM$ defined by,

$$\mathcal{U}^k = x_k^j \frac{\partial}{\partial x^j}, \quad 1 \leq k \leq m.$$

The metric g induces a positive-definite scalar product on $(\pi'^m)^* TM$ also denoted by g . From the definition of the open subset of Frenet jets $\mathcal{F}^{m-1}(M)$ in $J^{m-1}(\mathbb{R}, M)$ it follows that the system $(\mathcal{U}^1, \dots, \mathcal{U}^{m-1})$ is linearly independent on the open subset $\mathcal{F}^{m-1}(M) \times_M TM$ in $(\pi'^{m-1})^* TM$. By applying the Gram-Schmidt process on such open subset to the sections $(\mathcal{U}^1, \dots, \mathcal{U}^m)$, one obtains

the following systems of local sections $(\mathcal{X}_1, \dots, \mathcal{X}_m)$, $(\mathcal{Y}_1, \dots, \mathcal{Y}_m)$ of $(\pi'^m)^*TM$:

$$\begin{aligned}\mathcal{Y}_i &= \mathcal{U}^i - \sum_{h=1}^{i-1} g(\mathcal{U}^i, \mathcal{X}_h) \mathcal{X}_h, \quad 1 \leq i \leq m, \\ \mathcal{X}_i &= \frac{\mathcal{Y}_i}{|\mathcal{Y}_i|}, \quad 1 \leq i \leq m-1, \\ \mathcal{X}_m &= \mathcal{X}_1 \times \dots \times \mathcal{X}_{m-1},\end{aligned}$$

and according to [11, Theorem 4.2] one has $\varkappa_i = \frac{|\mathcal{Y}_{i+1}|}{|\mathcal{Y}_1| \cdot |\mathcal{Y}_i|}$, $1 \leq i \leq m-1$. From the very definitions, one deduces $\mathcal{U}^k|_{j^m\sigma} = U^{\sigma,k}$ (see (1)) and $\mathcal{X}_i|_{j^m\sigma} = X_i^\sigma$ (see Proposition 3.2). Hence for $2 \leq i \leq m$, we obtain

$$\begin{aligned}\frac{\partial \varkappa_{i-1}}{\partial x_i^j} &= \frac{1}{|\mathcal{Y}_1| \cdot |\mathcal{Y}_{i-1}|} \frac{\partial |\mathcal{Y}_i|}{\partial x_i^j}, \\ |\mathcal{Y}_i|^2 &= g(\mathcal{U}^i, \mathcal{U}^i) - \sum_{h=1}^{i-1} g(\mathcal{U}^i, \mathcal{X}_h)^2,\end{aligned}$$

and taking derivatives with respect to ∂x_i^j on the second formula, we have

$$\begin{aligned}|\mathcal{Y}_i| \frac{\partial |\mathcal{Y}_i|}{\partial x_i^j} &= x_i^j - \sum_{h=1}^{i-1} g(\mathcal{U}^i, \mathcal{X}_h) \mathcal{X}_h(x^j) \\ &= \left(\mathcal{U}^i - \sum_{h=1}^{i-1} g(\mathcal{U}^i, \mathcal{X}_h) \mathcal{X}_h \right)(x^j) \\ &= \mathcal{Y}_i(x^j).\end{aligned}$$

Therefore

$$k_j^i = \frac{\mathcal{Y}_i(x^j)}{|\mathcal{Y}_1| \cdot |\mathcal{Y}_{i-1}| \cdot |\mathcal{Y}_i|} (j_{t_0}^m \sigma), \quad 2 \leq i \leq m.$$

Hence

$$\begin{aligned}\det(k_j^i)_{i,j=1}^m &= \det \left(\frac{\mathcal{Y}_i(x^j)}{|\mathcal{Y}_1| \cdot |\mathcal{Y}_{i-1}| \cdot |\mathcal{Y}_i|} (j_{t_0}^m \sigma) \right)_{i,j=1}^m \\ &= \frac{1}{(|\mathcal{Y}_1|^{m+2} |\mathcal{Y}_m| \prod_{a=2}^{m-1} |\mathcal{Y}_a|^2) (j_{t_0}^m \sigma)} \cdot \begin{vmatrix} \mathcal{Y}_1(x^1) & \mathcal{Y}_1(x^2) & \dots & \mathcal{Y}_1(x^m) \\ \mathcal{Y}_2(x^1) & \mathcal{Y}_2(x^2) & \dots & \mathcal{Y}_2(x^m) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{Y}_m(x^1) & \mathcal{Y}_m(x^2) & \dots & \mathcal{Y}_m(x^m) \end{vmatrix} (j_{t_0}^m \sigma),\end{aligned}$$

and accordingly,

$$\det(k_j^i)_{i,j=1}^m = \frac{\det(X_1^\sigma(t_0), \dots, X_m^\sigma(t_0)) (\mathcal{Y}_1, \dots, \mathcal{Y}_m)}{|\mathcal{Y}_1|^{m+2} |\mathcal{Y}_m| \prod_{a=2}^{m-1} |\mathcal{Y}_a|^2} (j_{t_0}^m \sigma).$$

As the vectors $\mathcal{Y}_1, \dots, \mathcal{Y}_m$ are pairwise orthogonal its determinant is $\prod_{a=1}^m |\mathcal{Y}_a|$.

Hence

$$\begin{aligned} \det(k_j^i)_{i,j=1}^m &= \frac{1}{|\mathcal{Y}_1|^m \prod_{a=1}^{m-1} |\mathcal{Y}_a|} (j_{t_0}^m \sigma) \\ &= \frac{1}{\kappa_0^\sigma(t_0)^{\frac{m(m+1)}{2}} \kappa_1^\sigma(t_0)^{m-1} \kappa_2^\sigma(t_0)^{m-2} \dots \kappa_{m-2}^\sigma(t_0)}, \end{aligned}$$

which is positive. Hence $\lambda_{m-i}^{i-1} = 0$. \square

8.2 Few isometries

Below we consider Riemannian manifolds (M, g) such that, $\dim \mathfrak{i}(M, g) \leq \dim M$.

Proposition 8.2. *If (M, g) is a connected complete Riemannian manifold of class C^ω and $\mathfrak{i}(M, g)$ admits a basis (X_1, \dots, X_l) such that the tangent vectors $((X_1)_{x_0}, \dots, (X_l)_{x_0})$ at a point $x_0 \in M$ are linearly independent, then $N_r = mr + 1 + m - l$, $\forall r \geq 0$. The order of asymptotic stability of such manifolds is $r = 1$; namely, for every $r \geq 2$ a basis of differential invariants of order r can be obtained by applying the operator D_t to a basis of first-order invariants successively.*

Proof. According to the hypothesis of the statement, the set of points $x \in M$ for which $((X_1)_x, \dots, (X_l)_x)$ are linearly independent, is a dense open subset $O \subseteq M$, and for every $j_t^r \sigma \in J^r(\mathbb{R}, O)$ the homomorphism $(\pi_0^r)_*: \mathfrak{D}_{j_t^r \sigma}^r \rightarrow \mathfrak{D}_{\sigma(t)}^0$ is an isomorphism. The formula for N_r now follows from Lemma 5.3.

If $(t, I_i)_{i=l+1}^m$ is a basis of differential invariants of order 0, as $N_1 = 2m + 1 - l$, then there exist l new invariants of (strict) order 1, say (I_1, \dots, I_l) , such that $(t, I_{l+1}, \dots, I_m, D_t(I_{l+1}), \dots, D_t(I_m), I_1, \dots, I_l)$ is a basis of differential invariants of order 1. Actually, if

$$0 = \sum_{i=1}^{m-l} \lambda^i d(D_t(I_{l+i})) = \sum_{i=1}^{m-l} \lambda^i \left\{ x_1^k \frac{\partial^2 I_{l+i}}{\partial x^j \partial x^k} dx^j + \frac{\partial I_{l+i}}{\partial x^j} dx^j \right\},$$

then λ^i must vanish, as $\text{rk}(\partial I_{l+i} / \partial x^j) = m - l$. We can repeat this process indefinitely as in Theorem 5.11, taking account of the fact that the number of differential invariants of strict order r in a basis of invariants is $N_{r+1} - N_r = m$. \square

Remark 8.3. If $\dim \mathfrak{i}(M, g) = 1$ or 2, then $\mathfrak{i}(M, g)$ always admits a basis of linearly independent vector fields at some point $x_0 \in M$. In fact, if $X, \rho X, \rho \in C^\infty(M)$, are Killing vector fields for g , then $0 = L_{\rho X} g = d\rho \cdot c(X \otimes g)$, where the dot denotes symmetric product and $c: TM \otimes S^2 T^*M \rightarrow T^*M$ is the contraction operator. As the symmetric algebra of every cotangent space is an integral domain, it follows $c(X \otimes g) = 0$, but in this case ρX would be a Killing vector field for every ρ , which contradicts the fact that $\mathfrak{i}(M, g)$ is a finite-dimensional Lie algebra (e.g., see [20, VI, Theorem 3.3]).

Proposition 8.4 (cf. [15, (1.3)], [18, I.11, Theorem 1]). *Let (G, g) be a connected Lie group with Lie algebra \mathfrak{g} endowed with a left-invariant Riemannian metric. Two smooth curves $\sigma_1, \sigma_2: \mathbb{R} \rightarrow G$ are congruent with respect to G if and only if*

$$(44) \quad \omega(T^{\sigma_1}(t)) = \omega(T^{\sigma_2}(t)), \quad \forall t \in \mathbb{R},$$

where $\omega \in \Omega^1(G, \mathfrak{g})$ denotes the Maurer-Cartan form.

Proof. If there exists an element $\gamma \in G$ such that $\sigma_1 = L_\gamma \circ \sigma_2$, then

$$\begin{aligned} \omega(T^{\sigma_1}(t)) &= \omega((L_\gamma)_* T^{\sigma_2}(t)) \\ &= ((L_\gamma)^* \omega)(T^{\sigma_2}(t)) \\ &= \omega(T^{\sigma_2}(t)). \end{aligned}$$

Conversely, assume the equation (44) holds and let $\gamma \in G$ be the only element such that $\sigma_1(0) = \gamma \cdot \sigma_2(0)$. On the product manifold $\mathbb{R} \times G$, let Z be the vector field defined as follows: $Z_{(t, \alpha)} = (\partial/\partial t, \omega(T^{\sigma_i}(t))_\alpha)$, $\forall (t, \alpha) \in \mathbb{R} \times G$, $i = 1, 2$, where $\omega(T^{\sigma_i}(t))_\alpha$ denotes the evaluation of the left invariant vector field $\omega(T^{\sigma_i}(t)) \in \mathfrak{g}$ at $\alpha \in G$ for $i = 1, 2$; i.e.,

$$(45) \quad \omega(T^{\sigma_i}(t))_\alpha = \left. \frac{d}{du} \right|_{u=0} L_\alpha \circ \exp(u\omega(T^{\sigma_i}(t))).$$

Hence $\tilde{\sigma}_i(t) = (t, \sigma_i(t))$ is an integral curve of Z , $i = 1, 2$. Moreover, the curve $\tilde{\sigma}(t) = (t, (L_\gamma \circ \sigma_2)(t))$ is also an integral curve of Z . In fact, from (45) we obtain

$$\begin{aligned} T^{\tilde{\sigma}}(t) &= \left(\frac{\partial}{\partial t}, (L_\gamma)_* T^{\sigma_2}(t) \right) \\ &= \left(\frac{\partial}{\partial t}, \left. \frac{d}{du} \right|_{u=0} L_{\gamma \cdot \gamma_2(t)} \circ \exp(u\omega((L_\gamma)_* T^{\sigma_2}(t))) \right) \\ &= \left(\frac{\partial}{\partial t}, \left. \frac{d}{du} \right|_{u=0} L_{\gamma \cdot \gamma_2(t)} \circ \exp(u\omega(T^{\sigma_2}(t))) \right) \\ &= \left(\frac{\partial}{\partial t}, \left. \frac{d}{du} \right|_{u=0} L_{\gamma \cdot \gamma_2(t)} \circ \exp(u\omega(T^{\sigma_1}(t))) \right) \\ &= \left(\frac{\partial}{\partial t}, \omega(T^{\sigma_1}(t))_{\gamma \cdot \gamma_2(t)} \right). \end{aligned}$$

As $\tilde{\sigma}_1(0) = (0, \sigma_1(0)) = (0, \gamma \cdot \sigma_2(0)) = \tilde{\sigma}(0)$, from the uniqueness of a solution to a system of ordinary differential equations we conclude $\tilde{\sigma}_1(t) = \tilde{\sigma}(t)$, $\forall t$, i.e., $\sigma_1 = L_\gamma \circ \sigma_2$. \square

Corollary 8.5. *Let (G, g) be a connected Lie group with Lie algebra \mathfrak{g} endowed with a left-invariant Riemannian metric such that $\dim \mathfrak{I}(G, g) = \dim G$. Two smooth curves $\sigma_1, \sigma_2: \mathbb{R} \rightarrow G$ are congruent with respect to $\mathfrak{I}^0(G, g)$ if and only if (44) holds. Moreover, a basis of invariants of order $\leq r$ is $\{t, (D_t)^k(I)\}_{k=0}^{r-1}$, where $I = (I_1, \dots, I_m): J^1(\mathbb{R}, M) \rightarrow \mathfrak{g}$ are the functions $I(j_t^1 \sigma) = \omega(T^\sigma(t))$.*

Proposition 8.6. *If (M, g) is a complete Riemannian connected manifold of class C^ω such that $\mathfrak{i}(M, g)$ admits a basis (X_1, \dots, X_m) , $m = \dim M$, with linearly independent tangent vectors $((X_1)_{x_0}, \dots, (X_m)_{x_0})$ at $x_0 \in M$, then (M, g) is isometric to $(\mathfrak{I}^0(M, g), \bar{g})/H$, where H is a finite subgroup in $G = \mathfrak{I}^0(M, g)$, and \bar{g} is a left-invariant Riemannian metric.*

Proof. The mapping $\mu: G \rightarrow M$, $\mu(\phi) = \phi(x_0)$, is G -equivariant, i.e., $\gamma \circ \mu = \mu \circ L_\gamma$, where $\gamma \in G$ and L_γ denotes the left translation by γ . Hence μ is of constant rank and $\text{im } \mu$ is an open subset in M by virtue of the assumption, which we claim is the whole manifold M . In fact, if $x \in \partial(\text{im } \mu)$, then there exists a sequence $x_n = \gamma_n(x_0)$ such that $\lim x_n = x$; as the action of G is proper (see [31, 5.2.4]), we conclude that there is a convergent subsequence $\gamma_{n_k} \rightarrow \gamma$. Hence $x = \gamma(x_0)$ and the image of μ is a closed subset. Therefore the isotropy subgroup H of the point x_0 is a finite subgroup ([20, I, Corollary 4.8]) in G and $G/H \cong M$, where H acts on the right on G . The metric $\bar{g} = \mu^*(g)$ is invariant under left translations of elements in G ; in fact,

$$L_\gamma^*(\bar{g}) = (L_\gamma^* \circ \mu^*)(g) = (\gamma \circ \mu)^*(g) = \mu^*(\gamma^*(g)) = \mu^*(g) = \bar{g}.$$

□

Proposition 8.7. *Let $\omega: TG \rightarrow \mathfrak{g}$ be the Maurer-Cartan form of a connected Lie group G and let $\omega_h: TG \rightarrow \mathfrak{g}/H$ be map given by $\omega_h(X) = \omega(X) \bmod H$, $\forall X \in TG$, where H is a finite subgroup acting on the right on \mathfrak{g} by setting, $X \cdot h = \text{Ad}_{h^{-1}} X$, $\forall h \in H$, $\forall X \in \mathfrak{g}$, and Ad denoting the adjoint representation of G . As ω_h is H -invariant, it induces a “1-form” $\tilde{\omega}_h$ on $M \cong G/H$ with values in the quotient \mathfrak{g}/H (which is not a vector space). With the same hypotheses of Proposition 8.6 about (M, g) , two curves $\sigma_1, \sigma_2: (a, b) \rightarrow M$ are congruent under G if and only if, $\tilde{\omega}_h(T^{\sigma_1}(t)) = \tilde{\omega}_h(T^{\sigma_2}(t))$, $\forall t \in (a, b)$.*

Proof. The curve σ_i can be lifted to $\gamma_i: (a, b) \rightarrow G$, i.e., $\sigma_i(t) = \gamma_i(t) \bmod H$, for $i = 1, 2$, as $G \rightarrow M \cong G/H$ is a covering. The condition $\gamma(\sigma_1(t)) = \sigma_2(t)$, $\gamma \in G$, is equivalent to $\gamma \cdot \gamma_1(t) = \gamma_2(t) \cdot h$ for some $h \in H$. From Proposition 8.4 this last condition holds if and only if, $\omega(T^{\gamma_1}(t)) = \omega((R_h)_* T^{\gamma_2}(t))$, $\forall t$. As ω is left invariant, the previous equation is equivalent to the following:

$$\begin{aligned} \omega(T^{\gamma_1}(t)) &= \omega((L_{h^{-1}})_*(R_h)_* T^{\gamma_2}(t)) \\ &= \text{Ad}_{h^{-1}}(\omega(T^{\gamma_2}(t))), \end{aligned}$$

or equivalently, $\omega_h(T^{\gamma_1}(t)) = \omega_h(T^{\gamma_2}(t))$, thus concluding the proof. □

Remark 8.8. As H is finite, the adjoint representation of H on \mathfrak{g} is completely reducible and a classical result (see [26]) assures that the algebra of smooth invariants $C^\infty(\mathfrak{g})^H$ admits a finite basis, i.e., $C^\infty(\mathfrak{g})^H = C^\infty[I_1, \dots, I_k]$, I_1, \dots, I_k being H -invariant polynomial functions on \mathfrak{g} . Two curves σ_1, σ_2 are congruent if and only if, $I_i(\tilde{\omega}_h(T^{\sigma_1}(t))) = I_i(\tilde{\omega}_h(T^{\sigma_2}(t)))$ for $1 \leq i \leq k$; i.e., $\tilde{\omega}_h$ can be replaced by the ordinary scalar invariants $I_1 \circ \tilde{\omega}_h, \dots, I_k \circ \tilde{\omega}_h$.

For $\dim \mathfrak{J}^0(M, g) < \dim M$, as a consequence of the results in this section, we can state the following:

Proposition 8.9. *Assume (M, g) is a complete Riemannian connected manifold of class C^ω that satisfies the following two conditions: i) $\dim \mathfrak{J}^0(M, g) = l < m$, and ii) there exists a basis (X_1, \dots, X_l) for $\mathfrak{i}(M, g)$ such that the tangent vectors $((X_1)_{x_0}, \dots, (X_l)_{x_0})$ are linearly independent at $x_0 \in M$. We set $G = \mathfrak{J}^0(M, g)$.*

Let $S \subset M$ be a slice at x_0 , let U be a G -invariant neighbourhood of x_0 and let $r: U \rightarrow G \cdot x_0$ be a G -equivariant retraction of the inclusion $G \cdot x_0 \subset U$ such that, 1) $r^{-1}(x_0) = S$ (cf. [31, I, 5.1.11]), 2) the assumption ii) above holds for every $x \in S$. Then, $G_x = G_{x_0}$ for every $x \in S$, and we can define $\hat{\omega}_h: TU \rightarrow \mathfrak{g}/H$, with $H = G_{x_0}$, by setting, $\hat{\omega}_h(X) = \tilde{\omega}_h(\pi_x X)$, where $\tilde{\omega}_h: T(G/H) \rightarrow \mathfrak{g}/H$ is defined in Proposition 8.7, $\pi_x: T_x U \rightarrow T_x(G \cdot x)$ denotes the orthogonal g_x -projection and the canonical isomorphism $G \cdot x \cong G/H$ is used.

Let $q: U \rightarrow U/G \cong S/H$ be the quotient mapping. If $(\bar{x}^1, \dots, \bar{x}^{m-l})$ is a coordinate system on S/H , then $I_i = \bar{x}^i \circ q$ is a system of $m-l$ differential invariants of order zero on U . If $\sigma_1, \sigma_2: (a, b) \rightarrow M$ are two curves such that, $\sigma_1(t_0) = \sigma_2(t_0) = x_0$, and $\phi \circ \sigma_1 = \sigma_2$, for some $\phi \in G$, then

$$\text{a) } \hat{\omega}_h(T^{\sigma_1}(t_0)) = \hat{\omega}_h(T^{\sigma_2}(t_0)).$$

b) The invariants I_1, \dots, I_{m-l}, I , where $I: J^1(R, U) \rightarrow \mathfrak{g}/H$, defined by $I(j_t^1 \sigma) = \hat{\omega}_h(T^\sigma(t))$, are functionally independent.

c) The invariants $t, I_i, D_t I_i$, $1 \leq i \leq m-l$, and I are a basis of first-order differential invariants and they generate the ring of invariant functions of arbitrary order $r \geq 2$.

8.3 Surfaces

Theorem 8.10. *Let (M, g) be a Riemannian bidimensional manifold of class C^ω , let $\mathcal{O}^1 \subset J^1(\mathbb{R}, M)$ be the open subset of immersive jets, which coincides with Frenet curves in this case (see Remark 4.8), let (X_1^σ, X_2^σ) be the Frenet frame of an immersion σ , let K^g be the Gaussian curvature of g , and let ∇^g be the Levi-Civita connection of g . The functions $I_i^g: \mathcal{O}^1 \rightarrow \mathbb{R}$, $1 \leq i \leq 4$, defined by $I_1^g(j_{t_0}^1 \sigma) = g(T_{t_0}^\sigma, T_{t_0}^\sigma)$, $I_2^g(j_{t_0}^1 \sigma) = (dK^g)(X_1^\sigma)$, $I_3^g(j_{t_0}^1 \sigma) = (dK^g)(X_2^\sigma)$, and $I_4^g(j_{t_0}^1 \sigma) = (\nabla^g dK^g)(X_1^\sigma, X_1^\sigma)$ are invariant. If $\mathcal{M} \rightarrow M$ denotes the bundle of Riemannian metrics on M of class C^ω , then there exists a dense open subset $U^5 \subset J^5(\mathcal{M})$ such that for every metric g for which $j^5 g$ takes values in U^5 the invariants (I_1^g, \dots, I_4^g) are functionally independent and two Frenet curves $\sigma, \bar{\sigma}$ with values in (M, g) are congruent on some neighbourhoods of $x_0 = \sigma(t_0)$ and $\bar{x}_0 = \bar{\sigma}(t_0)$ if and only if, the following conditions hold:*

$$\kappa_0^\sigma(t) = \kappa_0^{\bar{\sigma}}(t), \quad \kappa_1^\sigma(t) = \kappa_1^{\bar{\sigma}}(t),$$

for small enough $|t - t_0|$, and

$$\begin{aligned} (dK^g)(X_1^\sigma)(x_0) &= (dK^g)(X_1^{\bar{\sigma}})(\bar{x}_0), \\ (dK^g)(X_2^\sigma)(x_0) &= (dK^g)(X_2^{\bar{\sigma}})(\bar{x}_0), \\ (\nabla^g dK^g)(X_1^\sigma, X_1^\sigma) &= (\nabla^g dK^g)(X_1^{\bar{\sigma}}, X_1^{\bar{\sigma}}). \end{aligned}$$

Proof. Let $\tilde{I}_i: \mathcal{O}^1 \rightarrow \mathbb{R}$, $0 \leq i \leq 4$, be the invariant functions defined as follows:

$$\begin{aligned}\tilde{I}_0(j_{t_0}^1 \sigma) &= t_0, \\ \tilde{I}_1^g(j_{t_0}^1 \sigma) &= \kappa_0^\sigma(t_0), \\ \tilde{I}_2^g(j_{t_0}^1 \sigma) &= \nabla^g R_4^g(X_1^\sigma, X_1^\sigma, X_2^\sigma, X_1^\sigma, X_2^\sigma)(\sigma(t_0)), \\ \tilde{I}_3^g(j_{t_0}^1 \sigma) &= \nabla^g R_4^g(X_2^\sigma, X_1^\sigma, X_2^\sigma, X_1^\sigma, X_2^\sigma)(\sigma(t_0)), \\ \tilde{I}_4^g(j_{t_0}^1 \sigma) &= (\nabla^g)^2 R_4^g(X_1^\sigma, X_1^\sigma, X_2^\sigma, X_1^\sigma, X_2^\sigma)(\sigma(t_0)),\end{aligned}$$

R_4^g being the Riemann-Christoffel tensor of g . In local coordinates $x = x^1$, $y = x^2$, $\dot{x} = x_1^1$, $\dot{y} = x_1^2$, we have

$$\begin{aligned}(46) \tilde{I}_1^g &= (I_1^g)^{\frac{1}{2}} = (g_{11}\dot{x}^2 + 2g_{12}\dot{x}\dot{y} + g_{22}\dot{y}^2)^{\frac{1}{2}}, \\ (47) \tilde{I}_2^g &= I_2^g = (I_1^g)^{-1} (\dot{x}K_x^g + \dot{y}K_y^g), \\ (48) \tilde{I}_3^g &= I_3^g = (I_1^g)^{-1} \det(g_{ij})^{-\frac{1}{2}} ((g_{11}\dot{x} + g_{12}\dot{y})K_y^g - (g_{12}\dot{x} + g_{22}\dot{y})K_x^g), \\ (49) \tilde{I}_4^g &= I_4^g = (I_1^g)^{-2} (\dot{x}^2 K_{xx}^g + 2\dot{x}\dot{y}K_{xy}^g + \dot{y}^2 K_{yy}^g),\end{aligned}$$

where $g = \sum_{i,j=1}^2 g_{ij} dx^i \otimes dx^j$, $g_{12} = g_{21}$, which proves that (I_1^g, \dots, I_4^g) are invariant functions according to Theorem 4.4, and also that $(\tilde{I}_0, \tilde{I}_1^g, \dots, \tilde{I}_4^g)$ are functionally independent if and only if $(I_1^g, \tilde{I}_2^g = \tilde{I}_1^g I_2^g, \tilde{I}_3^g = \tilde{I}_1^g I_3^g, \tilde{I}_4^g = (\tilde{I}_1^g)^2 I_4^g)$ are. Moreover, as a simple—although rather long—computation shows one has

$$\frac{\partial (I_1^g, \tilde{I}_2^g, \tilde{I}_3^g, \tilde{I}_4^g)}{\partial (x, y, \dot{x}, \dot{y})} (j_{t_0}^1 \sigma) = g(T_{t_0}^\sigma, T_{t_0}^\sigma) \det(g_{ij}(x_0))^{-\frac{3}{2}} \Phi(j_{t_0}^1 \sigma, j_{x_0}^5 g),$$

where Φ is a polynomial function of the form

$$\Phi(j_{t_0}^1 \sigma, j_{x_0}^5 g) = \Phi_{11}(j_{x_0}^5 g) \dot{x}(j_{t_0}^1 \sigma)^2 + \Phi_{12}(j_{x_0}^5 g) \dot{x}(j_{t_0}^1 \sigma) \dot{y}(j_{t_0}^1 \sigma) + \Phi_{22}(j_{x_0}^5 g) \dot{y}(j_{t_0}^1 \sigma)^2,$$

and U^5 is given by $(\Phi_{11})^2 + (\Phi_{12})^2 + (\Phi_{22})^2 > 0$.

As $\dim J^1(\mathbb{R}, M) = 5$, we can conclude by simply applying Corollary 4.5 because the rest of invariants depends functionally on $(\tilde{I}_0, \tilde{I}_1^g, \dots, \tilde{I}_4^g)$. \square

Remark 8.11. From the previous theorem it follows that the order of asymptotic stability of generic Riemannian surface is ≤ 2 , which agrees with Theorem 5.4, but the order of asymptotic stability is ≤ 1 , indeed. In fact, we can solve the equations (46)–(49) for x, y, \dot{x}, \dot{y} and taking derivatives with respect to D_t we conclude that κ_1^σ can be written as a function of the invariants t, I_1^g, \dots, I_4^g and their total derivatives.

8.4 3-dimensional manifolds

In this section, (M, g) denotes a 3-dimensional simply connected complete Riemannian manifold of non-constant curvature. Because of the dimensional gap [19, II, Theorem 3.2] (also see [21]), $\dim \mathfrak{I}(M, g) \leq 4$. If $\dim \mathfrak{I}(M, g) \leq 2$, then

not much more can be said about the structure of invariants of (M, g) in addition of what has been said in general in the section 8.2; on the other hand, if $\dim \mathfrak{I}(M, g) = 3$ or 4 and $\mathfrak{I}(M, g)$ acts transitively on M , then the structure of invariants can fully be determined, as explained below.

Theorem 8.12. *Let (M, g) be a 3-dimensional simply connected complete Riemannian manifold such that, i) $\dim \mathfrak{I}(M, g) = 4$ and ii) $\mathfrak{I}(M, g)$ acts transitively on M . There exists a global unitary vector field Z on M such that, $\phi \cdot Z = Z$ for every $\phi \in \mathfrak{I}^0(M, g)$ and the function $I_1: J^1(\mathbb{R}, M) \rightarrow \mathbb{R}$ given by $I_1(j_t^1 \sigma) = g(Z_{\sigma(t)}, T^\sigma(t))$, is a first-order differential invariant. A complete system of differential invariants is given as follows: t, \varkappa_0, I_1 , and \varkappa_1 .*

Proof. As is known (e.g., see [5], [6], [9]), any simply connected complete homogeneous 3-manifold with 4-dimensional isometry group, fibres as an 1-dimensional principal fibre bundle over a homogeneous 2-manifold of constant sectional curvature κ . The orthogonal distribution to the fibres is the kernel of a connection form the curvature of which has constant norm τ . Such a principal bundle is usually denoted by $(M_{\kappa, \tau}, g_{\kappa, \tau})$. In a certain fibred coordinate system (x, y, z) the metric reads as follows:

$$(50) \quad g_{\kappa, \tau} = \frac{1}{\left(1 + \frac{\kappa}{4}(x^2 + y^2)\right)^2} (dx^2 + dy^2) + \left(\frac{\tau}{1 + \frac{\kappa}{4}(x^2 + y^2)} (ydx - xdy) + dz \right)^2.$$

The lines parallel to the z axis are the fibers of $M_{\kappa, \tau}$. Moreover, the tangent vector field $Z = \partial/\partial z$ is unitary and more exactly is the unitary infinitesimal generator defined by the right action of the structure group on $M_{\kappa, \tau}$ as a principal bundle.

We claim that every $\phi \in \mathfrak{I}^0(M_{\kappa, \tau}, g_{\kappa, \tau})$ maps fibres to fibres. Working on the expression (50) of the metric $g_{\kappa, \tau}$, one can check that the following vector fields are a basis for $\mathfrak{i}(M_{\kappa, \tau}, g_{\kappa, \tau})$:

$$\begin{aligned} X_1 &= -2\kappa xy \frac{\partial}{\partial x} + (\kappa x^2 - \kappa y^2 - 4) \frac{\partial}{\partial y} + 4\tau x \frac{\partial}{\partial z}, \\ X_2 &= 2\kappa xy \frac{\partial}{\partial y} + (\kappa x^2 - \kappa y^2 + 4) \frac{\partial}{\partial x} + 4\tau y \frac{\partial}{\partial z}, \\ X_3 &= -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, \\ Z &= \frac{\partial}{\partial z}, \end{aligned}$$

As $[Z, X] = 0, \forall X \in \mathfrak{i}(M_{\kappa, \tau}, g_{\kappa, \tau})$, it follows $\phi \cdot Z = Z, \forall \phi \in \mathfrak{I}^0(M_{\kappa, \tau}, g_{\kappa, \tau})$. If $\sigma_i: (a, b) \rightarrow M_{\kappa, \tau}, i = 1, 2$, are two curves such that, $\sigma_1 = \phi \circ \sigma_2$ for certain $\phi \in \mathfrak{I}^0(M_{\kappa, \tau}, g_{\kappa, \tau})$, then $g_{\kappa, \tau}(Z_{\sigma_1(t)}, T^{\sigma_1}(t)) = g_{\kappa, \tau}(Z_{\sigma_2(t)}, T^{\sigma_2}(t)), \forall t \in (a, b)$. Hence the function I_1 in the statement is an invariant.

Moreover, from the formula for N_r in Lemma 5.3 (resp. Corollary 5.5) in this case one has $N_r = 3r + 4 - \text{rk } \mathfrak{D}^r|_{\mathcal{U}^r}$ (resp. $N_r = 3r, \forall r \geq 4$). As $\mathfrak{I}(M_{\kappa,\tau}, g_{\kappa,\tau})$ acts transitively on $M_{\kappa,\tau}$ one has, $\text{rk } \mathfrak{D}_{(t,x)}^0 = 3$, for all $(t, x) \in \mathbb{R} \times M_{\kappa,\tau}$; hence $N_0 = 1$, and the problem is reduced to compute $\text{rk } \mathfrak{D}^r$ for $r = 1, 2, 3$. From the formula (36), one obtains,

$$\begin{aligned} X_1^{(1)} &= -2\kappa xy \frac{\partial}{\partial x} + (\kappa x^2 - \kappa y^2 - 4) \frac{\partial}{\partial y} + 4\tau x \frac{\partial}{\partial z} \\ &\quad - 2\kappa (x_1 y + x y_1) \frac{\partial}{\partial x_1} + 2\kappa (x x_1 - y y_1) \frac{\partial}{\partial y_1} + 4\tau x_1 \frac{\partial}{\partial z_1}, \\ X_2^{(1)} &= 2\kappa xy \frac{\partial}{\partial y} + (\kappa x^2 - \kappa y^2 + 4) \frac{\partial}{\partial x} + 4\tau y \frac{\partial}{\partial z} \\ &\quad + 2\kappa (x_1 y + x y_1) \frac{\partial}{\partial y_1} + 2\kappa (x x_1 - y y_1) \frac{\partial}{\partial x_1} + 4\tau y_1 \frac{\partial}{\partial z_1}, \\ X_3^{(1)} &= -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} - y_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial y_1}, \\ Z^{(1)} &= \frac{\partial}{\partial z}. \end{aligned}$$

As a computation shows, one has $\text{rk } \mathfrak{D}^1|_{\mathcal{U}^1} = 4$ with $\mathcal{U}^1 = \{j_t^1 \sigma : T^\sigma(t) \neq 0\}$. As \mathfrak{D}^r is spanned by four vector fields, we conclude that $\text{rk } \mathfrak{D}^r \leq 4$, and since $\text{rk } \mathfrak{D}^r = \text{rk } \mathfrak{D}^{r-1} + \text{rk } \mathfrak{D}^{r,r-1}$ by virtue of (38), we finally obtain $\text{rk } \mathfrak{D}^r|_{\mathcal{U}^r} = 4$, $\forall r \geq 1$. Hence $N_r = 3r, \forall r \geq 1$.

For $r = 1$, the invariants are $t, \tilde{\varkappa}_0$, and I_1 , as they are functionally independent because of their expressions below,

$$\begin{aligned} \tilde{\varkappa}_0 &= \varkappa_0 - (I_1)^2 = \frac{(x_1)^2 + (y_1)^2}{\left(1 + \frac{\kappa}{4}(x^2 + y^2)\right)^2}, \\ I_1 &= \frac{\tau}{1 + \frac{\kappa}{4}(x^2 + y^2)} (y x_1 - x y_1) + z_1. \end{aligned}$$

Moreover, as a calculation shows, the curvature function $\varkappa_1 : \mathcal{F}^2(M) \rightarrow \mathbb{R}$ is given by,

$$\begin{aligned} (\varkappa_0)^6 (\varkappa_1)^2 &= (\varkappa_0)^2 ((g_{\kappa,\tau})_{11} (x_2)^2 + (g_{\kappa,\tau})_{22} (y_2)^2 + (g_{\kappa,\tau})_{33} (z_2)^2) \\ &\quad + 2(\varkappa_0)^2 ((g_{\kappa,\tau})_{12} x_2 y_2 + (g_{\kappa,\tau})_{13} x_2 z_2 + (g_{\kappa,\tau})_{23} y_2 z_2) \\ &\quad + A x_2 + B y_2 + C z_2 + \text{terms of order } \leq 1, \end{aligned}$$

A, B, C being homogeneous polynomials of degree 6 in the following functions:

- 1) $\varkappa_0, D_t \varkappa_0, x_1, y_1, z_1$,
- 2) the local coefficients $(g_{\kappa,\tau})_{1 \leq i \leq j \leq 3}$ of $g_{\kappa,\tau}$, and
- 3) the Christoffel symbols Γ_{ij}^h of the Levi-Civita connection of $g_{\kappa,\tau}$.

Following the proof of Theorem 5.11 and by using the formula (40), in the present case one has, $k_0 = N_0 - 1 = 0$, $k_1 = N_1 - 1 = 2$, $k_2 = N_2 - 1 - 2k_1 = 1$. Hence there are two functionally independent invariants of order 1, namely $\tilde{\varkappa}_0$,

I_1 , and one invariant of order 2, which is \varkappa_1 . We claim the invariants $t, \tilde{\varkappa}_0, I_1, D_t \tilde{\varkappa}_0, D_t I_1, \varkappa_1$ are generically functionally independent. In fact,

$$\begin{aligned} d\tilde{\varkappa}_0|_{\ker(\pi_0^1)_*} &= \frac{2(x_1\delta x_1 + y_1\delta y_1)}{(1 + \frac{\kappa}{4}(x^2 + y^2))^2}, \\ dI_1|_{\ker(\pi_0^1)_*} &= \frac{\tau(y\delta x_1 - x\delta y_1)}{1 + \frac{\kappa}{4}(x^2 + y^2)} + dz_1, \end{aligned}$$

$$\begin{aligned} d(D_t \tilde{\varkappa}_0)|_{\ker(\pi_1^2)_*} &= \frac{2(x_1\delta x_2 + y_1\delta y_2)}{(1 + \frac{\kappa}{4}(x^2 + y^2))^2}, \\ (\varkappa_0)^4 \varkappa_1 d\varkappa_1|_{\ker(\pi_1^2)_*} &= \{(g_{\kappa,\tau})_{11}x_2 + (g_{\kappa,\tau})_{12}y_2 + (g_{\kappa,\tau})_{13}z_2\}\delta x_2 \\ &\quad + \{(g_{\kappa,\tau})_{12}x_2 + (g_{\kappa,\tau})_{22}y_2 + (g_{\kappa,\tau})_{23}z_2\}\delta y_2 \\ &\quad + \{(g_{\kappa,\tau})_{13}x_2 + (g_{\kappa,\tau})_{23}y_2 + (g_{\kappa,\tau})_{33}z_2\}\delta z_2, \end{aligned}$$

where $\delta x_r = dx_r|_{\ker(\pi_{r-1}^r)_*}$, $\delta y_r = dy_r|_{\ker(\pi_{r-1}^r)_*}$, $\delta z_r = dz_r|_{\ker(\pi_{r-1}^r)_*}$ for $r = 1, 2$.

As $3 = \sum_{i=0}^3 k_i$, by virtue of (41), it follows $k_i = 0$, for $i > 2$; i.e., the subalgebra generated by the invariants of order < 3 , and their derivatives with respect to the operator D_t , exhausts the algebra of invariants for every order $i > 2$, and the given system is thus complete. \square

Theorem 8.13. *Let (M, g) be a 3-dimensional simply connected complete Riemannian manifold of class C^ω such that $\mathfrak{I}(M, g)$ is a 3-dimensional group acting transitively on M . Then (M, g) is isometric to $(\mathfrak{I}^0(M, g), \bar{g})$, \bar{g} being a left-invariant Riemannian metric. A complete system of invariants are: The function t and the three first-order invariants given by $I_i(j_i^1\sigma) = \omega_i(T^\sigma(t))$, $1 \leq i \leq 3$, $(\omega_1, \omega_2, \omega_3)$ being a basis for the Maurer-Cartan forms.*

Moreover, with the same notations and results as in [17], the metrics \bar{g} are the following:

- (i) *For the group $\widetilde{PSL}(2, \mathbb{R})$ (resp. $SU(2)$), $\bar{g} = \lambda(\omega_1)^2 + \mu(\omega_2)^2 + \nu(\omega_3)^2$, $\lambda, \mu, \nu \in \mathbb{R}^+$, with $\mu > \nu$ (resp. $\lambda > \mu > \nu$), where $(\omega_1, \omega_2, \omega_3)$ is the dual basis of a basis (X_1, X_2, X_3) of the algebra of left-invariant vector fields such that,*

$$\begin{aligned} [X_1, X_2] &= X_3, & [X_1, X_3] &= -X_2, & [X_2, X_3] &= -X_1, & G &= \widetilde{PSL}(2, \mathbb{R}), \\ [X_1, X_2] &= X_3, & [X_1, X_3] &= -X_2, & [X_2, X_3] &= X_1, & G &= SU(2). \end{aligned}$$

- (ii) *For the unimodular solvable group $\mathbb{R}^2 \rtimes_\varphi \mathbb{R}$, with coordinates (x, y, z) , and $\varphi(z) = \text{diag}(\exp(z), \exp(-z))$, $\omega_1 = \exp(-z)dx$, $\omega_2 = \exp(z)dy$, $\omega_3 = dz$,*
- $$\bar{g} = (\omega_1)^2 + (\omega_2)^2 + \nu(\omega_3)^2 \text{ or } \bar{g} = (\omega_1)^2 + \omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1 + \mu(\omega_2)^2 + \nu(\omega_3)^2,$$
- $\mu > 1, \nu > 0$.

- (iii) For the group $\tilde{E}_0(2) = \mathbb{C} \rtimes \mathbb{R}$, $\bar{g} = (\omega_1)^2 + \mu(\omega_2)^2 + \nu(\omega_3)^2$, $\mu, \nu \in \mathbb{R}^+$, with $0 < \mu < 1$, $\omega_1 = \cos(2\pi z)dx + \sin(2\pi z)dy$, $\omega_2 = \sin(2\pi z)dx - \cos(2\pi z)dy$, $\omega_3 = dz$, $x + iy \in \mathbb{C}$, $z \in \mathbb{R}$.
- (iv) For the non-unimodular solvable group $G_c = \mathbb{R}^2 \rtimes_{\varphi_c} \mathbb{R}$, with $c \neq 0$, and

$$\varphi_c(z) = \exp \begin{pmatrix} 0 & -cz \\ z & 2z \end{pmatrix} = \begin{pmatrix} a_1^1(z) & a_2^1(z) \\ a_1^2(z) & a_2^2(z) \end{pmatrix}, \quad z \in \mathbb{R},$$

$\omega_1 = a_1^1(-z)dx + a_2^1(-z)dy$, $\omega_2 = a_1^2(-z)dx + a_2^2(-z)dy$, $\omega_3 = dz$, we have

- (iv.a) If $c < 0$ or $c \geq 1$, then $\bar{g} = (\omega_1)^2 + \mu(\omega_2)^2 + \nu(\omega_3)^2$, $0 < \mu \leq |c|$, $\nu > 0$; if $c = 1$, the metrics $\bar{g} = (\omega_1)^2 + \lambda(\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1) + \nu(\omega_3)^2$, $0 < \lambda < 1$, $\nu > 0$, must also be included.
- (iv.b) If $0 < c < 1$, then

$$\begin{aligned} \bar{g} = & \frac{1}{2} \frac{(\mu+1)c-2}{c^2(c-1)} (\omega_1)^2 + \frac{1}{2} \frac{\mu-1}{c(c-1)} (\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1) \\ & + \frac{1}{2} \frac{\mu-1}{c-1} (\omega_2)^2 + \nu(\omega_3)^2, \end{aligned}$$

with $0 \leq \mu < 1$, $\nu > 0$.

Proof. According to Proposition 8.6 the Riemannian manifold (M, g) is isometric to $(G = \mathfrak{I}^0(M, g), \bar{g})$, \bar{g} being a left-invariant Riemannian metric and according to Proposition 8.7, t and I_1, I_2, I_3 constitute a complete system of invariants.

(i) Assume $G = \widetilde{PSL}(2, \mathbb{R})$ or $SU(2)$, and let $\mathfrak{X}^L(G)$ (resp. $\mathfrak{X}^R(G)$) be the Lie algebra of left (resp. right) invariant vector fields. As G is simple and \bar{g} is left invariant, we have (see [10, (5.1)Theorem], [30, Theorem 5]),

$$\mathfrak{X}^R(G) \subseteq \mathfrak{i}(G, \bar{g}) \subseteq \mathfrak{X}^L(G) + \mathfrak{X}^R(G).$$

As a calculation shows, $\mathfrak{X}^L(G) \cap \mathfrak{X}^R(G) = \{0\}$. Hence $\dim \mathfrak{i}(G, \bar{g}) \geq 4$ if and only if there exists a left-invariant Killing vector field $A \neq 0$. In this case $(R_{\exp(tA)} \circ L_{\exp(-tA)})^* \bar{g} = \bar{g}$, and taking derivatives,

$$(51) \quad \bar{g}(X, [A, Y]) + \bar{g}([A, X], Y) = 0, \quad \forall X, Y \in \mathfrak{X}^L(G).$$

We follow [28]. If $A = \sum_{i=1}^3 a_i X_i$, $a_i \in \mathbb{R}$, then the equation (51) is equivalent to the system $\sum_{i=1}^3 a_i (c_{ij}^h \lambda_h + c_{ih}^j \lambda_j) = 0$, $1 \leq h \leq j \leq 3$, $\bar{g} = \sum_{i=1}^3 \lambda_i (\omega_i)^2$, c_{jk}^i being the structure constants in the basis (X_1, X_2, X_3) . For $\widetilde{PSL}(2, \mathbb{R})$ the previous system becomes, $a_1(\lambda_2 - \lambda_3) = a_2(\lambda_3 + \lambda_1) = a_3(\lambda_1 + \lambda_2) = 0$; hence $a_2 = a_3 = 0$, and the value of a_1 is necessarily 0 if and only if $\lambda_2 \neq \lambda_3$. For $SU(2)$ the system is, $(\lambda_3 - \lambda_2)a_1 = (\lambda_3 - \lambda_1)a_2 = (\lambda_2 - \lambda_1)a_3 = 0$; hence $a_1 = a_2 = a_3 = 0$ for $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$.

(ii) If $G = \mathbb{R}^2 \rtimes_{\varphi} \mathbb{R}$, then it is readily seen that $\mathfrak{X}^R(G) = \langle Y_1, Y_2, Y_3 \rangle$, with $Y_1 = \partial/\partial x$, $Y_2 = \partial/\partial y$, $Y_3 = x\partial/\partial x - y\partial/\partial y + \partial/\partial z$. We know (see [17, Corollar

4.5]) that (G, \bar{g}) is not of constant curvature. If $\dim \mathfrak{i}(G, \bar{g}) = 4$, then according to Theorem 8.12 there exists a unitary vector field $Z = f\partial/\partial x + g\partial/\partial y + h\partial/\partial z$, $f, g, h \in C^\infty(x, y, z)$, in the center of $\mathfrak{i}(G, \bar{g})$. From $[Y_1, Z] = [Y_2, Z] = 0$, it follows that f, g , and h depend on z only, and by imposing $[Y_3, Z] = 0$, one obtains $f(z) = a \exp(z)$, $g(z) = b \exp(z)$, $h(z) = c$, with $a, b, c \in \mathbb{R}$, and the condition $L_Z \bar{g} = 0$ (for both classes of metrics in the statement) is easily proved to imply $a = b = c = 0$; hence $\mathfrak{i}(G, \bar{g})$ coincides with $\mathfrak{X}^R(G)$.

(iii) Similarly, in this case we have $\mathfrak{X}^r(\tilde{E}_0(2)) = \langle Y_1, Y_2, Y_3 \rangle$, with $Y_1 = \partial/\partial x$, $Y_2 = \partial/\partial y$, $Y_3 = 2\pi y \partial/\partial x - 2\pi x \partial/\partial y - \partial/\partial z$. As in the previous case, the assumption $\dim \mathfrak{i}(\tilde{E}_0(2), \bar{g}) = 4$ leads one to the following equations for the vector field Z : $f(z) = a \cos(2\pi z) + b \sin(2\pi z)$, $g(z) = a \sin(2\pi z) - b \cos(2\pi z)$, $h(z) = c$, with $a, b, c \in \mathbb{R}$. Taking account of the fact that $\mu \neq 1$, the condition $L_Z \bar{g} = 0$ then implies $a = b = c = 0$.

(iv.a) In this case, $\mathfrak{X}^r(G_c) = \langle Y_1, Y_2, Y_3 \rangle$, with $Y_1 = \partial/\partial x$, $Y_2 = \partial/\partial y$, $Y_3 = -cy\partial/\partial x + (x + 2y)\partial/\partial y + \partial/\partial z$. The assumption $\dim \mathfrak{i}(G_c, \bar{g}) = 4$ leads one to the following equations for the vector field Z : $f(z) = \alpha a_1^1(z) + \beta a_2^1(z)$, $g(z) = \alpha a_1^2(z) + \beta a_2^2(z)$, $h(z) = \gamma$, with $\alpha, \beta, \gamma \in \mathbb{R}$. By imposing $L_Z \bar{g} = 0$ for any one of the metrics $\bar{g} = (\omega_1)^2 + \mu(\omega_2)^2 + \nu(\omega_3)^2$, $0 < \mu \leq |c|$, $\nu > 0$, or $\bar{g} = (\omega_1)^2 + \lambda(\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1) + \nu(\omega_3)^2$, $0 < \lambda < 1$, $\nu > 0$, when $c = 1$, one concludes $Z = 0$.

(iv.b) This case follows by proceeding as in the previous one. \square

Remark 8.14. The other metrics in [17] admit a group of isometries of dimension either 4 or 6.

The results in the sections 7, 8.2, and 8.4 allow one to solve the equivalence problem completely in dimensions 2 and 3. In fact, if $\dim M = 2$, then $\dim \mathfrak{i}(M, g) \leq 3$; if $\dim \mathfrak{i}(M, g) \leq 2$, then the results in section 8.2 apply, and if $\dim \mathfrak{i}(M, g) = 3$, then (M, g) is of constant curvature and the classification follows from section 7. Similarly, if $\dim M = 3$, then the cases $\dim \mathfrak{i}(M, g) \leq 3$ also follow from section 8.2, the cases $\dim \mathfrak{i}(M, g) = 3$ or 4, with (M, g) Riemannian homogeneous, follow from section 8.4, and the manifold is of constant curvature in the case $\dim \mathfrak{i}(M, g) = 6$. The non-homogeneous case $\dim M = 3$ and $\dim \mathfrak{i}(M, g) = 4$ cannot occur as the isotropy subgroups of the action of $\mathfrak{I}(M, g)$ should be 2-dimensional tori, which is not possible. Moreover, the non-homogeneous case $\dim M = 3$ and $\dim \mathfrak{i}(M, g) = 3$ is further covered by the following

Proposition 8.15. *If (M, g) is a 3-dimensional Riemannian connected manifold such that $\mathfrak{i}(M, g) = \langle X_1, X_2, X_3 \rangle$ is 3-dimensional but $(X_1)_x, (X_2)_x, (X_3)_x$ are linearly dependent for every $x \in M$, then the orbits of the isometry group $G = \mathfrak{I}^0(M, g)$ on M are generically surfaces of constant signed curvature. Let U be a coordinate neighbourhood of a generic point $x_0 \in M$ with coordinates (x^1, x^2, x^3) , $x^i \in V^i \subset \mathbb{R}$, such that the foliation of the orbits of G intersected with U are given by $x^1 = \text{constant}$. Then x^1 is a differential invariant of order zero on U . If $\pi_x: T_x U \rightarrow T_x(G \cdot x)$ denotes the orthogonal g_x -projection,*

$x \in U$, then the function $\bar{\kappa}_0: J^1(\mathbb{R}, U) \rightarrow \mathbb{R}$ defined by $\bar{\kappa}_0(j_t^1 \sigma) = \kappa_0(\pi_x(\sigma'(t)))$ is a first-order differential invariant on U . The coordinates (x^1, x^2, x^3) induce a decomposition $J^2(\mathbb{R}, U) = J^2(\mathbb{R}, V^1) \times J^2(\mathbb{R}, V^2 \times V^3)$. Let π_2 be the projection onto the second factor.

The function $\bar{\kappa}_1: J^2(\mathbb{R}, U) \rightarrow \mathbb{R}$ defined by $\bar{\kappa}_1(j_t^2 \sigma) = \kappa_1(\pi_2(j_t^2 \sigma))$ is a second-order differential invariant on U , where κ_1 is the curvature function of the surfaces of the foliation.

Finally, the functions x^1 , $\bar{\kappa}_0$, and $\bar{\kappa}_1$ are a complete system of differential invariants on U .

The proof of the proposition above is similar to that of Proposition 8.9 and it will therefore be omitted.

We should also remark on the fact that the situation of the previous proposition can happen even in simple cases. For example, the isometry group of the metric

$$g = (1+x^2)dx^2 + (1+y^2)dy^2 + (1+z^2)dz^2 + xy(dx \otimes dy + dy \otimes dx) \\ + xz(dx \otimes dz + dz \otimes dx) + yz(dy \otimes dz + dz \otimes dy)$$

on \mathbb{R}^3 is $O(3)$.

References

- [1] D. V. Alekseevskij, A. M. Vinogradov, V. V. Lychagin, *Basic Ideas and Concepts of Differential Geometry* Encyclopaedia Math. Sci., 28, Springer, Berlin, 1991, pp. 1–264.
- [2] V. I. Arnold, *Mathematical Problems in Classical Physics*, Trends and Perspectives in Applied Mathematics, Appl. Math. Sci., Volume 100, Springer-Verlag, New York-Berlin, 1994, pp. 1–20.
- [3] L. Bérard Bergery, X. Chaurel, *A Generalization of Frenet's Frame for Non-degenerate Quadratic Forms with any index*, Séminaire de théorie spectrale et géométrie, Grenoble, vol. **20** (2002), 101–130.
- [4] A. L. Besse, *Manifolds all of whose geodesics are closed*, with appendices by D. B. A. Epstein, J.-P. Bourguignon, L. Bérard-Bergery, M. Berger and J. L. Kazdan, *Ergebnisse der Mathematik und ihrer Grenzgebiete* **93**, Springer-Verlag, Berlin-New York, 1978.
- [5] E. Cartan, *Leçons sur la Géométrie des Espaces de Riemann*, Gauthier-Villars, Éditeur, Paris, 1963.
- [6] B. Daniel, *Isometric immersions into 3-dimensional homogeneous manifolds*, Comment. Math. Helv. **82** (2007), no. 1, 87–131.
- [7] D. G. Ebin, *On the space of Riemannian metrics*, Bull. Amer. Math. Soc. **74** (1968), 1001–1003.

- [8] D. G. Ebin, *The manifold of Riemannian metrics*, Global Analysis (Proc. Sympos. Pure Math. , Volume XV, Berkeley, California, 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 11–40.
- [9] S. Engel, *On the geometry and trigonometry of homogeneous 3-manifolds with 4-dimensional isometry group*, Math. Z. **254** (2006), no. 3, 439–459.
- [10] C. Gordon, *Riemannian isometry groups containing transitive reductive subgroups*, Math. Ann. **248** (1980), no. 2, 185–192.
- [11] H. Gluck, *Higher curvatures of curves in euclidean space*, Amer. Math. Monthly **73** (1966), 699–704.
- [12] H. Gluck, *Higher curvatures of curves in euclidean space. II*, Amer. Math. Monthly **74** (1967), 1049–1056.
- [13] A. Gray, *The volume of a small geodesic ball of a Riemannian manifold*, Michigan Math. J. **20** (1973), 329–344 (1974).
- [14] M. L. Green, *The moving frame, differential invariants and rigidity theorems for curves in homogeneous spaces*, Duke Math. J. **45** (1978), no. 4, 735–779.
- [15] P. Griffiths, *On Cartan’s method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry*, Duke Math. J. **41** (1974), 775–814.
- [16] S. Gudmundsson, E. Kappos, *On the geometry of tangent bundles*, Expo. Math. **20** (2002), no. 1, 1–41.
- [17] K. Y. Ha, J. B. Lee, *Left invariant metrics and curvatures on simply connected three-dimensional Lie groups*, Math. Nachr. **282** (2009), no. 6, 868–898.
- [18] G. R. Jensen, *Higher Order Contact of Submanifolds of Homogeneous Spaces*, Lecture Notes in Math. **610**, Springer-Verlag, Berlin, 1977.
- [19] S. Kobayashi, *Transformation Groups in Differential Geometry*, Springer-Verlag, Berlin, Heidelberg, New York, 1972.
- [20] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, John Wiley & Sons, Inc. (Interscience Division), New York, Volume I, 1963; Volume II, 1969.
- [21] H. T. Ku, *Automorphism groups of some geometric structures*, J. Differential Geom. **15** (1980), 381–392.
- [22] R. S. Kulkarni, *Index theorems of Atiyah-Bott-Patodi and curvature invariants*, With a preface by Raoul Bott, Séminaire de Mathématiques Supérieures, no. 49 (Été 1972), Les Presses de l’Université de Montréal, Montreal, Que., 1975.

- [23] A. Kumpera, *Invariants Différentiels d'un Pseudogroupe de Lie I et II*, J. Differential Geometry **10** (1975), 289–345, 347–416.
- [24] J. M. Lee, *Introduction to Smooth Manifolds*, Graduate Text in Mathematics 218, Springer-Verlag, New York, 2003.
- [25] H. I. Levine, *Singularities of Differentiable Mappings*, Proceedings of Liverpool Singularities-Symposium I, Lecture Notes in Math. **192**, Springer-Verlag, Berlin, 1971, pp. 1–89.
- [26] D. Luna, *Fonctions différentiables invariantes sous l'opération d'un groupe réductif*, Ann. Inst. Fourier (Grenoble) **26** (1976), no. 1, ix, 33–49.
- [27] J. N. Mather, *Stability of C^∞ mappings: V, Transversality*, Advances in Math. **4** (1970), 301–336.
- [28] J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Advances in Math. **21** (1976), no. 3, 293–329.
- [29] K. Nomizu, *On Local and Global Existence of Killing Vector Fields*, Ann. of Math. **72** (1960), 105–120.
- [30] T. Ochiai, T. Takahashi, *The group of isometries of a left invariant Riemannian metric on a Lie group*, Math. Ann. **223** (1976), 91–96.
- [31] R. S. Palais, C.-L. Terng, *Critical point theory and submanifold geometry*, Lecture Notes in Math. **1353**, Springer-Verlag, Berlin, 1988.
- [32] M. Rochowski, *The Frenet frame of an immersion*, J. Differential Geometry **10** (1975), 181–200.
- [33] D. J. Struik, *Lectures on Classical Differential Geometry*, 2nd edition, Dover Publications, Inc., N. Y., 1961.
- [34] W. Thirring, *A Course in Mathematical Physics 2, Classical Field Theory*, Springer-Verlag, New York, 1979.
- [35] J-C Tougeron, *Idéaux de fonctions différentiables*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 71, Springer-Verlag, Berlin-New York, 1972.
- [36] J. A. Wolf, *Spaces of Constant Curvature*, McGraw-Hill Book Co., New York-London-Sydney, 1967.
- [37] K. Yamaguchi, *Geometrization of jet bundles*, Hokkaido Math. J. **12** (1983), no. 1, 27–40.
- [38] K. Yano, S. Ishihara, *Tangent and cotangent bundles: Differential Geometry*, Pure and Applied Mathematics, no. 16, Marcel Dekker, Inc., New York, 1973.